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Last time, I determined that an excellent approximation for the weakly-interacting array of paths was

$$\frac{1}{N} Log\left[PathCount(\alpha, N, D, P)\right] \approx \frac{P}{N}\log\binom{N}{\alpha N} - \frac{P-1}{N}\frac{\binom{N}{\alpha N+D}\binom{N}{\alpha N-D}}{\binom{N}{\alpha N}\binom{N}{\alpha N}}$$

The entropy density is given in “Phase Diagram of a Random Tiling Quasicrystal” as $\sigma(d) \approx d\log 2 - \frac{\pi^2}{23}d^3 + \ldots$, where d is the numeric density of rhombi: namely, to translate into the language of my approximation,

$$\sigma(d) = \frac{1}{N}\log PathCount\left(\frac{1}{2}, \infty, \frac{1}{d}, \infty d\right)$$

e.g., looking vertically up one row, I would expect the density of rhombi number-wise to be equal to $\frac{1}{separation} = d$. Now I recall that I wanted the area entropy density for comparison, and here I have the length-wise entropy density

$$Log\left[PathCount(\alpha, N, D, P)\right] \approx P\log\binom{N}{\alpha N} - (P-1)\frac{\binom{N}{\alpha N+D}\binom{N}{\alpha N-D}}{\binom{N}{\alpha N}^2}$$

$$Log\left[PathCount\left(\alpha = \frac{1}{2}, N, \frac{1}{d}, Nd\right)\right] \approx Nd\log\binom{N}{\alpha N} - (Nd-1)\frac{\binom{N}{\alpha N+\frac{1}{d}}\binom{N}{\alpha N-\frac{1}{d}}}{\binom{N}{\alpha N}^2}$$

$$\frac{1}{Area\left(\frac{1}{2}, N, D, P\right)} = \frac{1}{N^2 + \frac{1}{\sqrt{2}}(P\cdot N)} \to \frac{\sqrt{2}}{1+\sqrt{2}}\frac{1}{N^2}$$

$$\frac{\sqrt{2}}{1+\sqrt{2}}\frac{1}{N^2} Log\left[PathCount\left(\alpha = \frac{1}{2}, N, \frac{1}{d}, Nd\right)\right] \approx \frac{\sqrt{2}}{1+\sqrt{2}}\left[\frac{1}{N}d\log\binom{N}{\alpha N} - \frac{(Nd-1)}{N^2}\frac{\binom{N}{\alpha N+\frac{1}{d}}\binom{N}{\alpha N-\frac{1}{d}}}{\binom{N}{\alpha N}^2}\right]$$

So this is already looking somewhat close to what I want to see. Attempting to take a large-N limit, I have:

$$\frac{\sqrt{2}}{1+\sqrt{2}}\frac{1}{N^2}Log\left[PathCount\left(\alpha=\frac{1}{2},N\to\infty,\frac{1}{d},Nd\to\infty\right)\right]$$

$$\approx\frac{\sqrt{2}}{1+\sqrt{2}}\left[d\log\frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}-\frac{(Nd-1)}{N^2}\frac{\frac{N!}{\left(\alpha N+\frac{1}{d}\right)!\left((1-\alpha)N-\frac{1}{d}\right)!}\frac{N!}{\left(\alpha N-\frac{1}{d}\right)!\left((1-\alpha)N+\frac{1}{d}\right)!}}{\frac{N!}{\alpha N!(1-\alpha)N!}\frac{N!}{\alpha N!(1-\alpha)N!}}\right]$$

$$=\frac{\sqrt{2}}{1+\sqrt{2}}\left[d\log 2-\frac{(Nd-1)}{N^2}\frac{\alpha N!(1-\alpha)N!}{\left(\alpha N+\frac{1}{d}\right)!\left((1-\alpha)N-\frac{1}{d}\right)!}\frac{\alpha N!(1-\alpha)N!}{\left(\alpha N-\frac{1}{d}\right)!\left((1-\alpha)N+\frac{1}{d}\right)!}\right]$$

$$\approx\frac{\sqrt{2}}{1+\sqrt{2}}\left[d\log 2-\frac{d}{N}\frac{\alpha N!(1-\alpha)N!}{\left(\alpha N+\frac{1}{d}\right)!\left((1-\alpha)N-\frac{1}{d}\right)!}\frac{\alpha N!(1-\alpha)N!}{\left(\alpha N-\frac{1}{d}\right)!\left((1-\alpha)N+\frac{1}{d}\right)!}\right]$$

However, now I have two issues: one is to I to expand about. The small-density limit gives me a boring log 2, somehow missing the desired coefficient and the large-density limit goes against my original assumptions. To take the Stirling approximation would leave me with strange powers that I still can’t expand about (I tried both in Mathematica, both complained about the essential singularity and gave results in terms of the gamma function or otherwise non-polynomial functions).

I do see however that is recovers the proper first-order term (plus a constant factor related to the method by which the area is taken).

Suppose I had taken also the next-higher order term. Might this give me the desired behavior in P?

Looking at the approximation from last time, I see that I started with

$$\frac{1}{N}Log[PathCount(\alpha,N,D,P)]\approx$$

$$\frac{P}{N}\log\binom{N}{\alpha N}+\frac{1}{N}\sum_{p=2}^{P}\log\left(1+\sum_{i=2}^{p}(-1)^{i-1}\frac{\binom{N}{\alpha N+(i-1)D}\binom{N}{\alpha N-(i-1)D}}{\binom{N}{\alpha N}\binom{N}{\alpha N}}\right)$$

it is important to notice that in the weakly-interacting region, really only the first term in the innermost sum comes into play. Thus, I see that there is really no means to get any higher-order terms in P and therefore d since P only appeared only as an overall coefficient in the weakly-interacting case.