Ben Sauerwine
Condensed Matter
Progress Report for January 31, 2006

Proofs: Extend the long-term behavior of the entropy of one chain of 45-degree rhombi in an array of squares to multiple ones using the Gessel-Viennot algorithm.

The 1-chain example was done as a proof last week. I will do a couple of examples by hand and try to deduce a pattern, and possibly write a Mathematica program to solve it in the general case.

2 Chains:

Note that the Gessel-Viennot algorithm deals with only non-intersecting paths. To solve for non-crossing paths, a simple transformation on the starting and ending points exists where X should be increased by (index – 1) and Y decreased by (index – 1) for each point. Here I ignore the need for this transformation for my ultimate purpose of counting these 45-degree rhombus chains and assume that it has already been taken.

$$A = \det\begin{bmatrix} \binom{E_{1,x}-S_{1,x}+E_{1,y}-S_{1,y}}{E_{1,x}-S_{1,x}} & \binom{E_{1,x}-S_{2,x}+E_{1,y}-S_{2,y}}{E_{1,x}-S_{2,x}} \\ \binom{E_{2,x}-S_{1,x}+E_{2,y}-S_{1,y}}{E_{2,x}-S_{1,x}} & \binom{E_{2,x}-S_{2,x}+E_{2,y}-S_{2,y}}{E_{2,x}-S_{2,x}} \end{bmatrix}$$

The values of S are assumed to be given, as well as the long-term behavior of the endpoints E for each chain which will look like $E_{n,x} \to N\alpha_n + S_{n,x}, E_{n,y} \to N(1-\alpha_n) + S_{n,y}$.

$$A = \binom{E_{1,x}-S_{1,x}+E_{1,y}-S_{1,y}}{E_{1,x}-S_{1,x}}\binom{E_{2,x}-S_{2,x}+E_{2,y}-S_{2,y}}{E_{2,x}-S_{2,x}} - \binom{E_{1,x}-S_{2,x}+E_{1,y}-S_{2,y}}{E_{1,x}-S_{2,x}}\binom{E_{2,x}-S_{1,x}+E_{2,y}-S_{1,y}}{E_{2,x}-S_{1,x}}$$

Now since I want to eventually find
$\lim_{N\to\infty}\frac{1}{N}\ln A(N)$, I can take

$$A = \binom{N}{N\alpha_1}\binom{N}{N\alpha_2} - \binom{N\alpha_1 + S_{1,x} - S_{2,x} + N(1-\alpha_1) + S_{1,y} - S_{2,y}}{N\alpha_1 + S_{1,x} - S_{2,x}}\binom{N\alpha_2 + S_{2,x} - S_{1,x} + N(1-\alpha_2) + S_{2,y} - S_{1,y}}{N\alpha_2 + S_{2,x} - S_{1,x}}$$

So I see now that this only depends on the differences between the initial starting points and the long-term behavior of the 45-degree rhombus staircases. Let
$d_x = S_{2,x} - S_{1,x} \quad d_y = S_{2,y} - S_{1,y}$, and rewrite

$$A = \binom{N}{N\alpha_1}\binom{N}{N\alpha_2} - \binom{N - d_x - d_y}{N\alpha_1 - d_x}\binom{N + d_x + d_y}{N\alpha_2 + d_x}.$$

Or

$$A = \binom{N}{N\alpha_1}\binom{N}{N\alpha_2}\left[1 - \frac{\binom{N - d_x - d_y}{N\alpha_1 - d_x}\binom{N + d_x + d_y}{N\alpha_2 + d_x}}{\binom{N}{N\alpha_1}\binom{N}{N\alpha_2}}\right]$$

$$\ln A = \ln\left[\binom{N}{N\alpha_1}\right] + \ln\left[\binom{N}{N\alpha_2}\right] + \ln\left[1 - \frac{\binom{N - d_x - d_y}{N\alpha_1 - d_x}\binom{N + d_x + d_y}{N\alpha_2 + d_x}}{\binom{N}{N\alpha_1}\binom{N}{N\alpha_2}}\right]$$

Both terms in the expression above are on the same order. In fact, as the starting distance becomes smaller, A approaches zero. The first term gives the upper bound (which is not surprising, as it is the product of the total possible number of paths if there were no interaction at all), and the second term corrects for the starting positions. There is no good way to take the natural logarithm of this and then apply the Stirling approximation, so I need to find a relationship between the two that is a plausible function of the starting distance and then factor it out and treat it separately. The only real tool I have here is $\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$. This will certainly converge for all values I choose since the fraction must be less than one, and will likely converge rapidly as well.

$$\ln A = \ln\left[\binom{N}{N\alpha_1}\right] + \ln\left[\binom{N}{N\alpha_2}\right] - \frac{\binom{N - d_x - d_y}{N\alpha_1 - d_x}\binom{N + d_x + d_y}{N\alpha_2 + d_x}}{\binom{N}{N\alpha_1}\binom{N}{N\alpha_2}}$$

Simplifying, I have

$$\frac{\binom{N - d_x - d_y}{N\alpha_1 - d_x}\binom{N + d_x + d_y}{N\alpha_2 + d_x}}{\binom{N}{N\alpha_1}\binom{N}{N\alpha_2}}$$

$$= \frac{(N - d_x - d_y)!(N\alpha_1)!(N - N\alpha_1)!}{N!(N\alpha_1 - d_x)!(N - N\alpha_1 + d_x)!}\frac{(N + d_x + d_y)!(N\alpha_2)!(N - N\alpha_2)!}{N!(N\alpha_2 + d_x)!(N - N\alpha_2 + d_x)!}$$

This number must be less than one, and works particularly well for large separations and so naturally gives the expected value of the product of two 45-degree rhombus staircases for very large separations. e.g.,

$$\lim_{N \to \infty} \frac{1}{N} \ln\binom{AN + B}{CN + D} = A \ln A + (Y - A)\ln(A - Y) - Y \ln Y\ .$$

I was more interested in an approximation that would work for small ones, and clearly in order to do so I need a new approach to the natural logarithm. Unfortunately, the one identity that would allow its internal sum to be split up uses the hyperbolic arctangent which is in turn given by the natural logarithm of a sum.
http://functions.wolfram.com/ElementaryFunctions/Log/16/02/

Now as I write this, I have completed a Mathematica program to allow me to explore the long-term behavior of these overlaid streaks, including consideration of interaction through plotting.

Amazingly, even for relatively small separations the paths' long-term behavior seems to show that the entropy for the pair of paths is identical for that which would be expected for two paths that were totally independent! Thus, the next experiment I ran took paths that were parallel and varied initial separation distance to see what the effects totally attributable to initial separation were. Shockingly, even here initial separation had a negligible effect to the point where the only way to change the entropy per size dramatically was to set initial conditions to the not allowed case of "both paths start at the same point".

It would not be surprising to see this trend continue to larger quantities of paths, and so I'll be satisfied if it seems to be the same for the Gessel-Viennot algorithm for 3 paths. Indeed, in the final example in this week's companion notebook, it is clear that except in cases where the long-term behavior or the initial conditions require the paths to cross, the entropy of many paths in the long term is virtually identical to that which you'd expect from multiple independent paths. There's one more experiment I'd like to run: what if I had a "Gambler's Ruin" situation with several parallel paths? Let's find out.

I call it "Gambler's Ruin" because with 3 paths and one going down the center, I effectively limit only the center path by following parallel long-term behaviors on all three. Looking at the last example, it's clear that except when the final path is extraordinarily close to or on one of the other two paths, its entropy in the long term is roughly indifferent to the its relative starting position, even when the ultimate results are parallel.

Conclusions:

In the long term, except where the paths would be forced to cross, the entropy of the long-term behavior of multiple non-intersecting paths given by the Gessel-Viennot algorithm can be approximated even in small separations by the sum of the entropies of that many paths following their respective slopes independently of one another!