

Progress Report Feb. 28, 2006  
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## **Gessel-Viennot Algorithm Proof Notes**

### **What is the Gessel-Viennot Algorithm?**

The premise of the Gessel-Viennot algorithm is simple: Given a number of starting points and ending points (each ending point corresponding to a particular starting point), if a line is drawn along the lattice on a Cartesian grid going only right and down, find the number of non-touching arrangements of paths (two paths touch if at any time they occupy the same lattice point) that can be drawn.

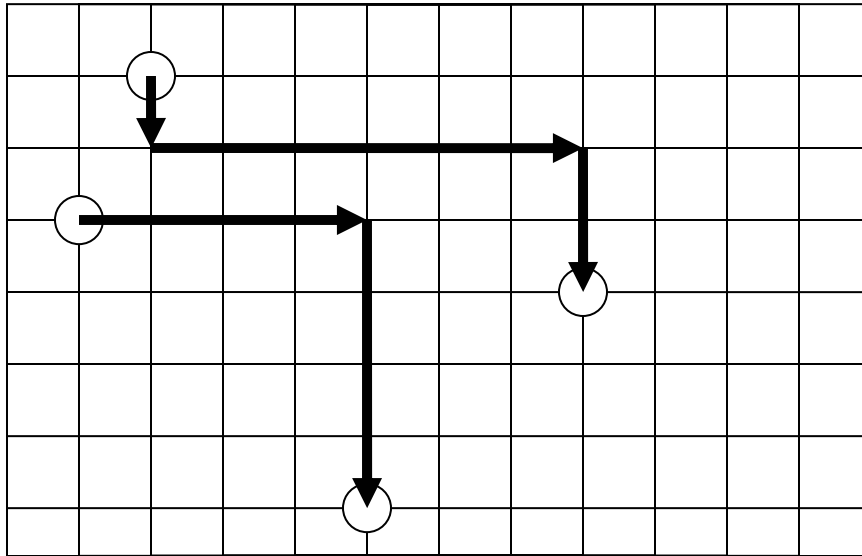
Note that any starting points and ending points given must satisfy the condition that if the ending points corresponding to any two starting points are swapped, then their corresponding drawn paths must also cross.

### **Basis Case and Justification**

Consider two starting points, and two corresponding endpoints. They are arranged such that if a line is drawn between start point 1 and the end point 2, it is guaranteed to touch any path drawn from the start point 2 to end point 1 (in fact, this is always the case for two paths.)

Further, if any binomial has a negative parameter in any Gessel-Viennot matrix, let it instead be zero. This technicality is a hindrance to general derivations involving the Gessel-Viennot algorithm.

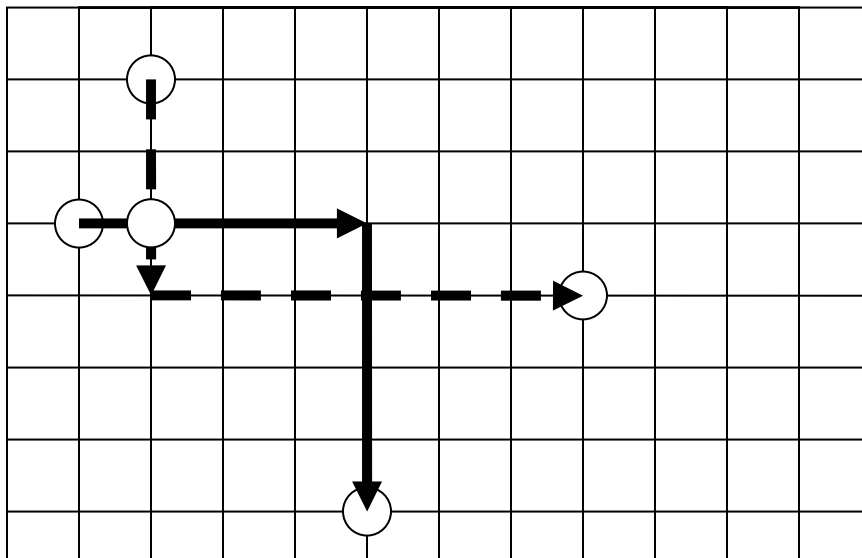
Now consider the two paths earlier: certainly, they must either touch somewhere, or not touch:



Here, the paths do not touch. Using Pascal's Triangle, one can see that the total number of paths from a start  $(S_x, S_y)$  point to an end point  $(E_x, E_y)$  is  $\binom{E_x - S_x + E_y - S_y}{E_x - S_x}$ , where the indexing increases going down and to the right. Then, the total number of possible paths ignoring interactions that can be drawn is:

$$\binom{E_{x,1} - S_{x,1} + E_{y,1} - S_{y,1}}{E_{x,1} - S_{x,1}} \binom{E_{x,2} - S_{x,2} + E_{y,2} - S_{y,2}}{E_{x,2} - S_{x,2}}$$

Now suppose that at some time, the two paths touched. I consider only the first point where this occurred.



Based on this, it's clear that if I started at the top starting point, I could "jump" onto the leftmost starting point's solid path and follow it the rest of the way to the bottom

endpoint. In the same way, I could “jump” onto the dashed path from the solid path at the touch point and follow the dashed path all the way to the rightmost end point.

Now I see that with this point of first contact, I have a generated exactly one selection of paths from Point 1’s starting point to Point 2’s ending point, and from Point 2’s starting point to Point 1’s ending point (the definition that I “jump” tracks at the first touch point only ensures this). Further, this is a bijective mapping due to the fact that “jumping” tracks twice would result in the exact same original configuration. Thus, in order to get rid of all paths that have touched at least once (and therefore these must have a first touch point), I must subtract:

$$\begin{pmatrix} E_{x,2} - S_{x,1} + E_{y,2} - S_{y,1} \\ E_{x,2} - S_{x,1} \end{pmatrix} \begin{pmatrix} E_{x,1} - S_{x,2} + E_{y,1} - S_{y,2} \\ E_{x,1} - S_{x,2} \end{pmatrix}$$

Combining these, I get the determinant for a two-by-two matrix:

$$\begin{pmatrix} E_{x,1} - S_{x,1} + E_{y,1} - S_{y,1} \\ E_{x,1} - S_{x,1} \end{pmatrix} \begin{pmatrix} E_{x,2} - S_{x,2} + E_{y,2} - S_{y,2} \\ E_{x,2} - S_{x,2} \end{pmatrix} - \begin{pmatrix} E_{x,2} - S_{x,1} + E_{y,2} - S_{y,1} \\ E_{x,2} - S_{x,1} \end{pmatrix} \begin{pmatrix} E_{x,1} - S_{x,2} + E_{y,1} - S_{y,2} \\ E_{x,1} - S_{x,2} \end{pmatrix}$$

Corresponding to

$$Det \begin{bmatrix} S_1 \rightarrow E_1 & S_1 \rightarrow E_2 \\ S_2 \rightarrow E_1 & S_2 \rightarrow E_2 \end{bmatrix}. \text{ In fact, in general the total number of arrangements of paths}$$

possible is the determinant of the square matrix

$$Det \left[ [S_N \rightarrow E_M]_{NM} \right]$$

assuming that the condition given in the earlier section is satisfied.

### **Inclusion-Exclusion: n-Paths**

I have to give more consideration to how to simplify this argument to the simplest-explanation level.

### **Derivation of Periodic/Long Path Case**

(See companion Mathematica notebook GVLlongterm7.nb)

Consider the expression from the February 21 progress report:

$$\frac{1}{N} \text{Log} [\text{PathCount}(\alpha, N, D, P)] \approx \frac{P}{N} \log \binom{N}{\alpha N} + \frac{1}{N} \sum_{p=2}^P \log \left( 1 + \sum_{i=2}^p (-1)^{i-1} \frac{\binom{N}{\alpha N + (i-1)D} \binom{N}{\alpha N - (i-1)D}}{\binom{N}{\alpha N} \binom{N}{\alpha N}} \right)$$

Now I'd like to let  $\{N, P\} \rightarrow \infty$ . First, I'd like to confirm the plausibility of allowing them both to go to infinity together, linearly. If the approximation becomes more accurate as they go together to infinity, that will be sufficient support for this approach.

The results from the Mathematica notebook indicate that if they go to infinity together with N and P linearly the result diverges. It indicates that allowing  $N \propto P^2$  or higher ensures convergence. Now I seek:

$$\lim_{P \rightarrow \infty} \left[ \frac{P}{P^2} \log \binom{P^2}{\alpha P^2} + \frac{1}{P^2} \sum_{p=2}^P \log \left( 1 + \sum_{i=2}^p (-1)^{i-1} \frac{\binom{P^2}{\alpha P^2 + (i-1)D} \binom{P^2}{\alpha P^2 - (i-1)D}}{\binom{P^2}{\alpha P^2} \binom{P^2}{\alpha P^2}} \right) \right]$$

$$= \lim_{P \rightarrow \infty} \left[ \frac{1}{P^2} \sum_{p=1}^P \log \left( \sum_{i=1}^p (-1)^{i-1} \binom{P^2}{\alpha P^2}^{-1} \binom{P^2}{\alpha P^2 + (i-1)D} \binom{P^2}{\alpha P^2 - (i-1)D} \right) \right]$$

I expect the result for  $D = 1$  to be the expected result for one path alone,  $\text{Log}[2]$ .

In a previous progress report, I found that these elements of the sum inside the logarithm are constantly decreasing, and each less than one. Perhaps I can find a generating function for the parameter to the logarithm. Writing in terms of path length N and iterating parameter i above, I have inside the logarithm:

$$F(N) = \sum_{i=1}^p (-1)^{i-1} \binom{N}{\alpha N}^{-1} \binom{N}{\alpha N + (i-1)D} \binom{N}{\alpha N - (i-1)D}$$

$$= \Gamma(\alpha N) \Gamma((1-\alpha)N) \Gamma(N) \sum_{i=1}^p \frac{(-1)^{i-1}}{\Gamma(\alpha N + (i-1)D) \Gamma(\alpha N - (i-1)D) \Gamma((1-\alpha)N - (i-1)D) \Gamma((1-\alpha)N + (i-1)D)}$$

Most immediately, when I try to find a generating function for this, the reason why I had to select  $N = P^2$  becomes clear: this sum does not converge if D if the  $(i-1)D$  is allowed to consistently overpower the  $\alpha N$  in the gamma functions.

So what I need, essentially, is a function such that:

$$\left[ \left( \frac{\partial}{\partial x} \right)^i F \right] (1) = \frac{(-1)^i}{\Gamma(\alpha N + iD) \Gamma(\alpha N - iD) \Gamma((1-\alpha)N - iD) \Gamma((1-\alpha)N + iD)} \text{ so}$$

that I can collapse this into a generating function which I can then operate on with my logarithm.

The generalized regularized hypergeometric function nearly fits this bill:

$${}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (\alpha_j)_k z^k}{k! \prod_{j=1}^q \Gamma(\beta_j + k)}$$

$$q \geq p \vee q = p - 1 \wedge |k| < 1 \vee q = p - 1 \wedge |k| = 1 \wedge \operatorname{Re}\left(\sum_{j=1}^{p-1} \beta_j - \sum_{j=1}^p \alpha_j\right) > 0$$

This nearly gives me the basic form I need. I note that the definition of the Pochhammer symbol is:

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \quad ; \quad (\neg (-\alpha \in \mathbb{Z} \wedge -\alpha \geq 0 \wedge n \in \mathbb{Z} \wedge n \leq -\alpha))$$

I'll also need to use

$$\Gamma(nz + b) = n^{nz + b - \frac{1}{2}} (2\pi)^{\frac{1-n}{2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{b+k}{n}\right) \quad ; \quad n \in \mathbb{N}^+$$

This will remove the coefficients from my index variable, but then I'll have to find a way to deal with the new dependence on it outside the product.

Letting  $D=1$ , I get

$$\left[ \left( \frac{\partial}{\partial x} \right)^i F \right] (0) = \frac{(-1)^i}{\Gamma(\alpha N + i) \Gamma(\alpha N - i) \Gamma((1 - \alpha)N - i) \Gamma((1 - \alpha)N + i)}$$

Now in order to obtain a result of this form, I need to take:

$$b_1 = N\alpha \quad b_2 = (1 - \alpha)N$$

to get the “-i” terms in the denominator, I need an identity for  $\frac{1}{\Gamma(x - i)}$ .

Let me try:

$$\Gamma(z - n) = \frac{(-1)^n \Gamma(z)}{(1 - z)_n} \quad ; \quad n \in \mathbb{Z}$$

$$\Gamma(\alpha N - i) = \frac{(-1)^i \Gamma(\alpha N)}{(1 - \alpha N)_i} = \frac{1}{\Gamma(\alpha N - i)} = \frac{(1 - \alpha N)_i}{(-1)^i \Gamma(\alpha N)}$$

So for the  $D=1$  case, my generating function takes the form

$$F = \frac{1}{\Gamma(\alpha N) \Gamma((1 - \alpha)N)^2} F_2(1 - (1 - \alpha)N, 1 - \alpha N; (1 - \alpha)N, \alpha N; -z)$$

Letting  $z$  be 1 in this function gives a series

```

In[201]:= Clear[a]
FullSimplify[
Series[
  1
  Gamma[a N] Gamma[(1 - a) N]
HypergeometricPFQRegularized[{1 - (1 - a) N, 1 - a N}, {a N, (1 - a) N}, -z],
{z, 0, 3}]]
Out[202]=
1
Gamma[a N]^2 Gamma[N - a N]^2
z
Gamma[-1 + N - a N] Gamma[1 + N - a N] Gamma[-1 + a N] Gamma[1 + a N]
z^2
2 Gamma[-2 + N - a N] Gamma[2 + N - a N] Gamma[-2 + a N] Gamma[2 + a N]
z^3
6 (Gamma[-3 + N - a N] Gamma[3 + N - a N] Gamma[-3 + a N] Gamma[3 + a N]) + O[z]^4

```

which clearly matches my desired result. Recombining this and combining this with the expression above where I factored out  $\Gamma(\alpha N) \Gamma((1 - \alpha)N) \Gamma(N)$ , I have

$$= \lim_{P \rightarrow \infty} \left[ \frac{1}{P^2} \sum_{p=1}^P \log \left( \Gamma(P^2) \sum_{i=0}^{p-1} \frac{\partial^i}{\partial z^i} {}_2F_2(1 - (1 - \alpha)P^2, 1 - \alpha P^2; (1 - \alpha)P^2, \alpha P^2; -z) \right) \right]$$

$$= \lim_{P \rightarrow \infty} \frac{1}{P^2} \left[ P \log \Gamma(P^2) + \sum_{p=1}^P \log \left( \sum_{i=0}^{p-1} \frac{\partial^i}{\partial z^i} {}_2F_2(1 - (1 - \alpha)P^2, 1 - \alpha P^2; (1 - \alpha)P^2, \alpha P^2; -z) \right) \right]$$

where  $z \rightarrow 0$

Now I can write:

$$\frac{\partial^i {}_2F_2(a_1, \dots, a_p; b_1, \dots, b_q; z)}{\partial z^i} = \left( \prod_{j=1}^p (a_j)_i \right) {}_2F_2(a_1 + i, \dots, a_p + i; b_1 + i, \dots, b_q + i; z) \quad i, n \in \mathbb{N}^+$$

But certainly this would be a step in the wrong direction: this only undoes what I've worked so hard to do! Let me write

$$\sum_{i=1}^p \frac{\partial^i}{\partial z^i} {}_2F_2(1 - (1 - \alpha)P^2, 1 - \alpha P^2; (1 - \alpha)P^2, \alpha P^2; -z)$$

where  $z \rightarrow 0$

is roughly the same as the first p terms of

$${}_3F_2(1, 1 - (1 - \alpha)P^2, 1 - \alpha P^2; (1 - \alpha)P^2, \alpha P^2; -1)$$

and since these terms in the sum were constantly decreasing, this is fairly accurate within an error function. Let me see how closely:

```
In[351]:= Clear[a, P]
a = 1/3;
P = 4;
N[1/Gamma[a P] Gamma[(1 - a) P] HypergeometricPFQRegularized[{1, 1 - (1 - a) P, 1 - a P}, {a P, (1 - a) P}, -1]]
N[Sum[(-1)^i / (Gamma[a P + i] Gamma[a P - i] Gamma[(1 - a) P - i] Gamma[(1 - a) P + i]), {i, 0, 1}]]
N[Sum[(-1)^i / (Gamma[a P + i] Gamma[a P - i] Gamma[(1 - a) P - i] Gamma[(1 - a) P + i]), {i, 0, 3}]]
N[Sum[(-1)^i / (Gamma[a P + i] Gamma[a P - i] Gamma[(1 - a) P - i] Gamma[(1 - a) P + i]), {i, 0, 100}]]
Out[354]= 0.46306
Out[355]= 0.467415
Out[356]= 0.463079
Out[357]= 0.46306
```

This is a very good approximation, even after only a few terms!

$$\approx \lim_{P \rightarrow \infty} \frac{1}{P^2} \left[ P \log \Gamma(P^2) + \sum_{p=1}^P \log \left( {}_3F_2(1, 1 - (1 - \alpha)P^2, 1 - \alpha P^2; (1 - \alpha)P^2, \alpha P^2; -1) \right) \right] \text{Now,}$$

with the dependence on the summation p in the logarithm gone, I have

$$= \lim_{P \rightarrow \infty} \frac{1}{P^2} \left[ P \log \Gamma(P^2) {}_3F_2(1, 1 - (1 - \alpha)P^2, 1 - \alpha P^2; (1 - \alpha)P^2, \alpha P^2; -1) \right]$$

$$= \lim_{P \rightarrow \infty} \frac{1}{P} \left[ \log \Gamma(P^2) {}_3F_2(1, 1 - (1 - \alpha)P^2, 1 - \alpha P^2; (1 - \alpha)P^2, \alpha P^2; -1) \right]$$

Checking this versus the “Result” function which gave my approximation pre-simplification, I have:

```

In[515]:= Clear[a, P]
a = 1/2;
P = 50;
N[1/P^2 (P Log[Gamma[P^2] HypergeometricPFQRegularized[{1, 1 - (1 - a) P^2, 1 - a P^2}, {a P^2, (1 - a) P^2}, -1]])] - Result[0.5, 50^2, 1, 50]

a = 1/2;
P = 10;
N[1/P^2 (P Log[Gamma[P^2] HypergeometricPFQRegularized[{1, 1 - (1 - a) P^2, 1 - a P^2}, {a P^2, (1 - a) P^2}, -1]])] -
Result[0.5, 10^2, 1, 10]

Out[518]= 0.135858
Out[521]= 0.343265

```

These results show that indeed, this 3F2 approximation does indeed approach my desired value in the limit. However, this still diverges at infinity, and so cannot give me my desired result for the periodic case. It seems that because I allowed N and P to go to infinity as  $N = P^2$ , I allowed a lot of possible interactions to escape me. Maybe if I tried it the other way, it would work better than previous results indicated. However, even the original approximation doesn't work well here. In fact, its accuracy gets worse as N and P get larger together (see the companion notebook for this). Overall, it looks like while my approximation works for the finite case, it is not a valid expression for the determinant of the infinite matrix. Somehow, all three of these results fail to capture the spirit of the infinite-path count case. In the interest of simplification, however, I notice that I can almost simplify this further:

$${}_{q+1}P_q(1, a_2, \dots, a_{q+1}; a_2 + 1, \dots, a_{q+1} + 1; -1) = \frac{1}{2 \prod_{j=2}^{q+1} \Gamma(a_j)} \sum_{k=2}^{q+1} \left( \psi\left(\frac{a_k + 1}{2}\right) - \psi\left(\frac{a_k}{2}\right) \right) \prod_{\substack{l=2 \\ l \neq k}}^{q+1} \frac{1}{a_l - a_k} k$$

$$a_k \neq a_l \wedge 2 \leq l \leq q+1 \wedge 2 \leq k \leq q+1 \wedge l \neq k$$

In fact, my values for a only differ by a sign. The sum here might take a simpler form given the similarity of my coefficients to one another.

However, the point here is that this approximation cannot work: Letting P become large while holding N fixed isn't even a possibility, because the approximation works well for N larger than P. In fact, when N is smaller than P, whenever the binomials get a negative coefficient they should be replaced with zero in the Gessel-Viennot algorithm.

What this means is that in my infinite matrix, the columns would become almost identical except that the ordering of elements would have to change, e.g:

	To Endpoint	0	"-D"	"+D"	"-2D"	"+2D"
From		↓	↓	↓	↓	↓
StartPoint	0 →	()	()	()	0	0
StartPoint	"-D" →	()	()	0	()	0
StartPoint	"+D" →	()	0	()	0	()
StartPoint	"-2D" →	0	()	0	()	0

where the above is an example for of how ordering would complicate the matrix form, assuming that they truncate by  $-2D$ , e.g. this might be a case where the separation is four, alpha is one-half, and path-length is six.

However, trying to see the result converge as path count grows and length stays fixed, as I see in the Mathematica notebook, also does not yield anything close to convergence. It seems that the Gessel-Viennot algorithm is fundamentally unfit for periodic boundaries. The issue seems to be that no matter how many additional paths one adds, the additional room provided by the ends propagates along the entire array.