Ben Sauerwine
Progress Report Feb 21

Problem Setup:

Propose that I have P paths, of length N and periodic separation across the initial diagonal D. What is the long-term limit of their entropy per unit size as N becomes very large?

Explicit Counting:

The forms of the matrix in the Gessel-Viennot algorithm will look like

$$\begin{bmatrix} R_{11} & R_{12} & \ldots & R_{1P} \\ R_{21} & R_{22} & \ldots & R_{2P} \\ \ldots & \ldots & \ldots & \ldots \\ R_{P1} & R_{P2} & \ldots & R_{PP} \end{bmatrix}$$

The next step, then, is to decide how many paths exist to each endpoint. Due to the fact that each has a separation D from its neighbors (from the initial problem statement), I next impose that $N \geq (P-1)D$ so that each of these routings is non-vanishing.

Now each routing will take a value given by

$$R_{AB} = \binom{\alpha N + (A-1)D - (B-1)D + (1-\alpha)N - (A-1)D + (B-1)D}{\alpha N + (A-1)D - (B-1)D} = \binom{N}{\alpha N + (A-B)D}$$

Notice that from my $N \geq (P-1)D$ assumption, both of these elements will always be positive.

The next step is to determine the form of the determinant. I recall from my proof of the Gessel-Viennot algorithm that I have formed this using permutations of these elements using the inclusion-exclusion principle. Perhaps in examining this closely I may find a nice form for each inclusion and exclusion step, thereby finding the largest, second largest, and third largest contributors to the inevitable natural logarithm.

I recall that these are produced with 0-swap, 1-swap, 2-swap, …, P-1-swap permutations.

$Paths \times Permutation \quad count$

	0	1	2	3
1	1	0	0	0
2	1	1	0	0
3	1	3	2	0
4	1	6	11	6

This is going to be a count that is important to collapsing the inclusion-exclusion principle terms. Let me develop a recursion:

Define $Count[P,S]$ be the number of distinct permutations formable by swapping P paths' endpoints S times, where this path is first formable using exactly S swaps.

To develop a recursion, I certainly notice that $Count[P,0]=1$, which is obvious. There is only one distinct permutation formable with zero swaps. Certainly, $Count[P,1]=\binom{P}{2}$, since I may choose two elements to permute and swap them. It seems that $Count[P,P-1]=(P-1)!$, which makes sense since for each element I choose a new partner that has not been chosen yet and is not it (the last element will be determined already by the time I reach it). Applying this to the next row gives

1 10 35 50 24

$$\binom{5}{0} \quad \binom{5}{2} \quad \binom{5}{4}\left[\frac{1}{2}\binom{4}{2}+\binom{4}{3}2!\right] \quad \binom{5}{4}3!+\binom{5}{3}2! \quad 4!$$

justify these as such:

<u>Row 1</u>: There is nothing to choose, no permutation is made.

<u>Row 2:</u> Select two and swap them.

<u>Row 3:</u> Select four of the five elements. Of these four, I want to swap two and then swap the remaining two, but since order doesn't matter I only want half of them.
The other option would be to select three of these four and permute them. There are only two of them that can be arranged and keep orderings unique.

<u>Row 4:</u> I can either select four of the five to permute, giving me three of these to arrange, or select three of five to permute leaving two to arrange, and swap the other two.

<u>Row 5:</u> Select all five of the elements, leaving me four free to permute.

I'm not clear on how to generate these coefficients, but let $Count[P,S]$ nonetheless represent them.

Now perhaps I can find an expression giving the combinatorial result for one-swaps, two-swaps, et cetera in terms of the routing values, I can then generally address the result inside the natural logarithm and speak meaningfully of the most significant, second most significant, etc. path contribution.

Certainly, the zero-contribution will be:

$$R_{11}R_{22}R_{33}R_{44}....R_{PP} = \left[\binom{N}{\alpha N}\right]^P$$

For a one-swap, I'll be subtracting

$$R_{11}R_{22}...R_{AB}...R_{BA}...R_{PP} = \left[\binom{N}{\alpha N}\right]^{P-2}\binom{N}{\alpha N+(A-B)D}\binom{N}{\alpha N+(B-A)D}$$

$$=\left[\binom{N}{\alpha N}\right]^{P-2}\sum_{A=1}^{P-1}\sum_{B=A}^{P}\binom{N}{\alpha N+(A-B)D}\binom{N}{\alpha N+(B-A)D}$$

Things will only get more complex for more swaps, and so no easy relation is likely to appear. Further, by this procedure I will find myself dramatically under and over-counting results unless I can get all of them in the determinant.

Another Direction: Educated Guessing

Let me try looking at the problem from another direction. Of course a determinant for P paths will have $P!$ elements, and if I'm going to match the number of elements, I'll need to guess that it looks something like $(A)(A+B)(A+B+C+...)$, etc. Starting, I see I can certainly write the one and two path cases easily:

$$P=1 \quad \binom{N}{\alpha N}$$

$$P=2 \quad \binom{N}{\alpha N}\left[\binom{N}{\alpha N}-\binom{N}{\alpha N+D}\frac{\binom{N}{\alpha N-D}}{\binom{N}{\alpha N}}\right]$$

Now can I detect a pattern if I extend it to a 3 by 3 determinant? Consider:

$$R_{11}R_{22}R_{33}+R_{12}R_{23}R_{31}+R_{13}R_{21}R_{32}-R_{31}R_{22}R_{13}-R_{32}R_{23}R_{11}-R_{33}R_{21}R_{12}$$

Denoting these by the D term in the combinations, these look like

$$(0)(0)(0)+(D)(D)(-2D)+(2D)(-D)(-D)-(-2D)(0)(2D)-(-D)(D)(0)-(0)(-D)(D)$$

$$P=3 \quad \binom{N}{\alpha N}\left[\binom{N}{\alpha N}-\frac{\binom{N}{\alpha N-D}\binom{N}{\alpha N+D}}{\binom{N}{\alpha N}}\right]\left[\binom{N}{\alpha N}-\frac{\binom{N}{\alpha N+D}\binom{N}{\alpha N-D}}{\binom{N}{\alpha N}}+X\right]$$

$$(D)(D)(-2D)+(2D)(-D)(-D)-(-2D)(0)(2D)$$

Here the second term in the third expansion now leaves me with one additional term, unaccounted for. Also, the X in the third term cannot possibly provide all three of the remaining terms. However, this might well be a good approximation, which I will check. Also, a pattern starts becoming clear:

$$P=3 \quad \binom{N}{\alpha N}\left[\binom{N}{\alpha N}-\frac{\binom{N}{\alpha N-D}\binom{N}{\alpha N+D}}{\binom{N}{\alpha N}}\right]\left[\binom{N}{\alpha N}-\frac{\binom{N}{\alpha N+D}\binom{N}{\alpha N-D}}{\binom{N}{\alpha N}}+\frac{\binom{N}{\alpha N+2D}\binom{N}{\alpha N-2D}}{\binom{N}{\alpha N}}\right]$$

$$(D)(D)(-2D)+(2D)(-D)(-D)-(-2D)(0)(2D)$$

(See GVLongterm6.nb for numerical data vouching for the quality of this approximation.)

So, my approximation written explicitly is:

$$PathCount(\alpha,N,D,P)\approx\binom{N}{\alpha N}^{-P}\prod_{p=1}^{P}\left(\sum_{i=1}^{p}(-1)^{i-1}\binom{N}{\alpha N+(i-1)D}\binom{N}{\alpha N-(i-1)D}\right)$$

This is an expression that I may be able to work with. I can certainly break up my product easily with the logarithm, and then identifying the largest, second largest, etc. terms is simple since as the inner summation index increases, the value of the particular term will decrease.

The Long-Term Limit of my Educated Guess

There is no question that as $D\to 2N$, $PathCount(\alpha,N,D,P)\to PathCount(\alpha,N,D,1)^{P}$, the value $\frac{1}{N}Log[PathCount(\alpha,N,D,1)]$ which was found in the progress report dated January 24 (for 1 path, D takes no meaning.)

$$\frac{1}{N}Log[PathCount(\alpha,N,D,P)]\approx$$

$$-\frac{P}{N}\log\binom{N}{\alpha N}+\frac{1}{N}\sum_{p=1}^{P}\log\left(\sum_{i=1}^{p}(-1)^{i-1}\binom{N}{\alpha N+(i-1)D}\binom{N}{\alpha N-(i-1)D}\right)$$

Clearly, taking only the $i=1$ terms gives exactly the result for the paths where touching or crossing is not considered at all.

Now since my sums are already sorted ascending, I may take:

$$Log[Big+Small]=Log[Big]+Log\left[1+\frac{Small}{Big}\right]$$

$$\approx Log[Big]+\frac{Small}{Big}-\frac{1}{2}\left(\frac{Small}{Big}\right)^{2}+\frac{1}{3}\left(\frac{Small}{Big}\right)^{3}-\ldots$$

So now collapsing the largest contributions, I get

$$\frac{1}{N}Log[PathCount(\alpha,N,D,P)]\approx$$

$$\frac{P}{N}\log\binom{N}{\alpha N}+\frac{1}{N}\sum_{p=2}^{P}\log\left(1+\sum_{i=2}^{p}(-1)^{i-1}\frac{\binom{N}{\alpha N+(i-1)D}\binom{N}{\alpha N-(i-1)D}}{\binom{N}{\alpha N}\binom{N}{\alpha N}}\right)$$

Again, the first term here represents the limit ignoring any interaction between paths.

Using the result from January 7, as well as Mathematica, I take $N\to\infty$ and get

```
In[63]:=
Clear[a, d];
Limit[ (Binomial[N, a N + d] Binomial[N, a N - d]) / (Binomial[N, a N] Binomial[N, a N]),
 N → ∞]
Out[64]= 1
```

$$\frac{1}{N} Log\left[PathCount(\alpha, N, D, P)\right] \approx$$

$$P \ln\left(\frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right) + \frac{1}{N}\sum_{p=2}^{P} \log\left(1 + \sum_{i=2}^{p}(-1)^{i-1}(approaching \quad 1 \quad from \quad below)\right)$$

to justify the collapsing of the logarithm, I note that this is a non-issue for odd choices of P as

$$\frac{1}{N} Log\left[PathCount(\alpha, N, D, P)\right] \approx$$

$$P \ln\left(\frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right) + \frac{1}{N}\sum_{p=2}^{P} \log(1) = P \ln\left(\frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right) + \frac{1}{N}\sum_{p=2}^{P} 0 = P \ln\left(\frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right)$$

for even choices of P, I justify this by noting that since the value inside the logarithm approaches 1 from below, the argument remains overall positive and that as a logarithm bounded above by something that approaches a constant, it cannot approach its target value more quickly than the $\frac{1}{N}$ will send its contribution to zero. Further confirming this, I have

```
In[1]:= Limit[
  1/N Log[1 - (Binomial[N, a N + d] Binomial[N, a N - d]) / (Binomial[N, a N] Binomial[N, a N])],
  N → ∞]
Out[1]= 0
```

Conclusion:

Based on this approximation, I do not see any reason why the long-term limit of entropy per unit for any number of paths with any initial separation should deviate from the non-interacting case. With periodic conditions, however, such an argument seems plausible and will likely make a good research problem.