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Progress Report April 4, 2006

Building off of last week's report, I seek to confirm the distance-cubed dependence in the weakly interacting case for infinitely long paths. Certainly, I have a form that works but is extraordinarily difficult to integrate. However, it seems plausible that I might be able to take a first-order expansion on the hypergeometric function and get a form which I can integrate. Look at GVLongterm13 to see a comparison of the behavior of the approximation versus the actual.

What the first two experiments in that notebook show me is:

It looks like as distance grows, the approximation is good where $N < 2D^2$. However, this isn't particularly helpful: if I want to take the large-N limit, then I certainly cannot also let D go to infinity.

If I expand the F function in lambda polynomially, the tails will become less significant as N grows and more significant as D grows. It seems possible that in the large-N limit, the coefficients on larger values of lambda would shrink more quickly and a first-order expansion would suffice. I found that:

$$\ln(f_N(\lambda)) \approx \ln(A + B\lambda^2)$$

$$A(N,D,\alpha) = \binom{N}{\alpha N} + \frac{(-1)^D 2\Gamma(1+N)\Gamma(D-N+N\alpha)\sin(\pi(1+N-N\alpha))}{\pi\Gamma(D+1+N\alpha)} \times$$

$$F\left(\{1\} \cup \bigcup_{i=1}^{D}\left\{1+\frac{i-1}{D}-\frac{N}{D}+\frac{N\alpha}{D}\right\}; \bigcup_{i=1}^{D}\left\{1+\frac{i}{D}+\frac{N\alpha}{D}\right\}, (-1)^D\right)$$

$$B(N,D,\alpha) = \frac{\Gamma(1+N)\Gamma(D-N+N\alpha)\sin(\pi(1+N-N\alpha))}{\pi\Gamma(D+1+N\alpha)A(N,D,\alpha)} \times$$

$$\left(\begin{array}{l}(-1)^{D-1} F\left(\{1\} \cup \bigcup_{i=1}^{D}\left\{1+\frac{i-1}{D}-\frac{N}{D}+\frac{N\alpha}{D}\right\}; \bigcup_{i=1}^{D}\left\{1+\frac{i}{D}+\frac{N\alpha}{D}\right\}, (-1)^D\right) + \\ -3\prod_{i=1}^{D}\frac{\Gamma\left(1+\frac{i-1}{D}-\frac{N}{D}+\frac{N\alpha}{D}\right)}{\Gamma\left(1+\frac{i}{D}+\frac{N\alpha}{D}\right)} F\left(\{2\} \cup \bigcup_{i=1}^{D}\left\{2+\frac{i-1}{D}-\frac{N}{D}+\frac{N\alpha}{D}\right\}; \bigcup_{i=1}^{D}\left\{2+\frac{i}{D}+\frac{N\alpha}{D}\right\}, (-1)^D\right) + \\ (-1)^{D-1}\prod_{i=1}^{2D}\frac{\Gamma\left(1+\frac{i-1}{D}-\frac{N}{D}+\frac{N\alpha}{D}\right)}{\Gamma\left(1+\frac{i}{D}+\frac{N\alpha}{D}\right)} F\left(\{3\} \cup \bigcup_{i=1}^{D}\left\{3+\frac{i-1}{D}-\frac{N}{D}+\frac{N\alpha}{D}\right\}; \bigcup_{i=1}^{D}\left\{3+\frac{i}{D}+\frac{N\alpha}{D}\right\}, (-1)^D\right)\end{array}\right)$$

The result of the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(A+B\lambda^2\right)d\lambda =$$

$$\frac{1}{\pi}\left(2\sqrt{\frac{A(N,D,\alpha)}{B(N,D,\alpha)}}ArcTan\left(\pi\sqrt{\frac{B(N,D,\alpha)}{A(N,D,\alpha)}}\right)+\pi\left(-2+\ln A(N,D,\alpha)+\ln\left(1+\frac{B(N,D,\alpha)}{A(N,D,\alpha)}\pi^2\right)\right)\right)$$

does indeed agree closely with the expected results of both the actual integral of the generating function as well as the matrix determinant, but only when $N < 2D^2$.

The moral seems to be that these high-order terms in the expansion simply cannot be thrown out in the long-term limit (any more than, as shown in an earlier progress report, that I may simply take matrix entries to be entries in this limit). Again, the form of the integral to solve is:

$$\frac{1}{2\pi}\int\ln\left(A+B\left[e^{-i\lambda}F\left(c,e^{-i\lambda}\right)+e^{i\lambda}F\left(c,e^{i\lambda}\right)\right]\right)d\lambda$$

$$\int F(c,z)dz = g(c-1)F(c-1,z)$$

where F is the form of the hypergeometric function as it occurs here. Suppose that instead I took the limit as N becomes large, then integrated. Perhaps it would bring my integral to a nicer form. Recall that N appears only in my hypergeometric formulas. From last week, I had

$$f_N(\lambda,D)=t_{N,0}-4(-1)^{D-1}\left[\sin\left(\pi(1+N-N\alpha)\right)\frac{\Gamma(1+N)\Gamma(D-N+N\alpha)}{4\pi\Gamma(D+1+N\alpha)}\right]\cos(\lambda)\times$$

$$\left(\begin{array}{l} e^{-i\lambda}\,{}_{D+1}F_D\left(\{1\}\cup\bigcup_{j=1}^{D}\left\{\frac{D+j-1}{D}-\frac{N}{D}+\frac{N\alpha}{D}\right\};\bigcup_{j=1}^{D}\left\{\frac{D+j}{D}+\frac{N\alpha}{D}\right\};(-1)^De^{-i\lambda}\right)+ \\ e^{i\lambda}\,{}_{D+1}F_D\left(\{1\}\cup\bigcup_{j=1}^{D}\left\{\frac{D+j-1}{D}-\frac{N}{D}+\frac{N\alpha}{D}\right\};\bigcup_{j=1}^{D}\left\{\frac{D+j}{D}+\frac{N\alpha}{D}\right\};(-1)^De^{i\lambda}\right) \end{array}\right)$$

$$t_{N,0}=\binom{N}{\alpha N}$$

Recall that the hypergeometric functions take the form

$$F(\{A\},\{B\},z)=\sum_{n=0}^{\infty}\frac{\prod_{a\in A}\frac{\Gamma(a+n)}{\Gamma(a)}}{\prod_{b\in B}\frac{\Gamma(b+n)}{\Gamma(b)}}\frac{z^n}{n!}\qquad F(1\cup\{A\},\{B\},z)=\sum_{n=0}^{\infty}\frac{\prod_{a\in A}\frac{\Gamma(a+n)}{\Gamma(a)}}{\prod_{b\in B}\frac{\Gamma(b+n)}{\Gamma(b)}}z^n$$

$${}_DF_D(\{A\},\{B\},z)\sum_{n=0}^{\infty}\prod_{i=1}^{D}\frac{\Gamma(a_i+n)\Gamma(b_i)}{\Gamma(b_i+n)\Gamma(a_i)}z^n$$

Now I see that the source of the problem is that I have taken the original problem (a logarithm of a sum, with non-vanishing elements of similar size) and changed it into the integral of a logarithm of a sum, again with non-vanishing elements of similar size.

Indeed, examining this in the large-N limit does indicate that not indicate any simple pattern of annihilation.

It is interesting to note that by the form of this formula, when z = 1 (I integrate about the $|z|=1$ annulus), regardless of how large the separation N becomes, the index n will eventually dwarf it. By drawing a plot, the source of the discrepancy from just the binomial case is clear:

In[519]:=

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b = 900.5;
a = -1000.5;
ListPlot[Table[{n, Gamma[a + n]/Gamma[a] Gamma[b]/Gamma[b + n] (-1)^n}, {n, 0, 200}]]
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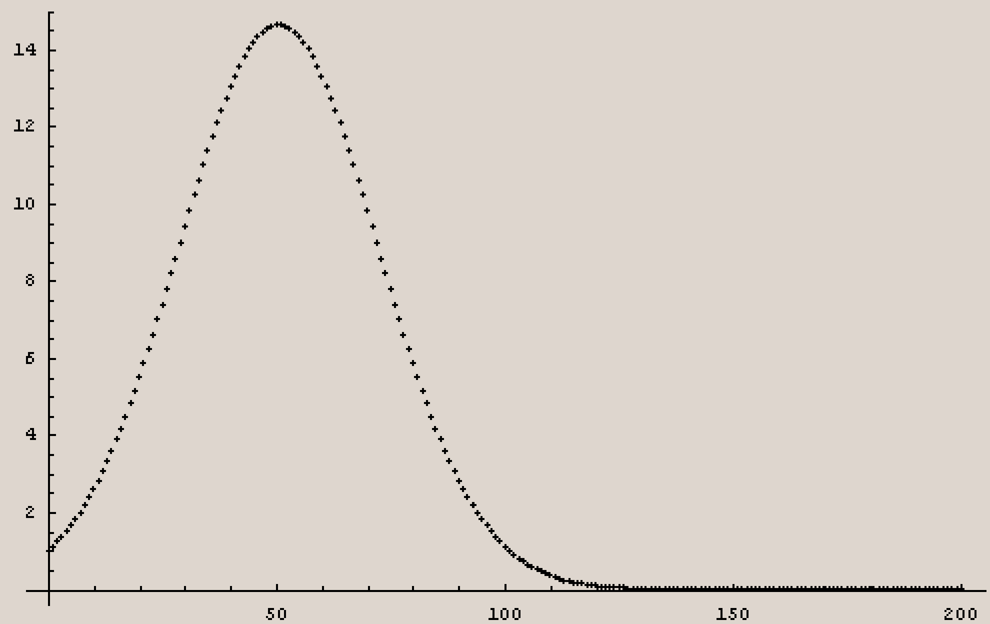

Out[521]= - Graphics -

As n grows, the coefficients grow, peak, and then level out. If I could exploit this distribution, perhaps I could find a way to collapse the sum in the large-n limit. I wonder if as n grows this function becomes pointy in such a way as to make it look like a delta-function with coefficient equal to the magnitude of its value at zero. In a way, to say this is a bit irresponsible: I should multiply the function's Fourier transform by that of the delta-function, but I have already seen the folly of the low-order Fourier route.