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In the last progress report I found a relatively tame form yielding the generating function for the Toeplitz matrix, but I couldn't sum the series. Today, I will try to collapse this sum using the Euler-MacLaurin formula.

e.g., $\sum_{k=1}^{n-1} f_k = \int_0^n f(k)dk - \frac{1}{2}\left[f(0) + f(n)\right] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right]$

I certainly hope that the Bernoulli number part goes away—I would like to deal with as few terms as possible in the result for ease of taking the logarithm.

Adapting this to my sum

$f_N(\lambda, D) = (\alpha N)^{-2\alpha N} N^N e^{\frac{-\lambda^2}{4k}} \sum_{x=-\infty}^{\infty} e^{-k\left(x - \frac{i\lambda}{2k}\right)^2}$, I expect

$$\sum_{k=-\infty}^{\infty} f_k = \int_{-\infty}^{\infty} f(k)dk - \frac{1}{2}\left[f(-\infty) + f(\infty)\right] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(\infty) - f^{(2k-1)}(-\infty)\right]$$

Since my sum has a nice Gaussian form and goes to zero in both limits, it seems plausible that I could just replace this with

$$k = \frac{D^2}{\alpha N}$$

$$\sum_{k=-\infty}^{\infty} f_x = (\alpha N)^{-2\alpha N} N^N e^{\frac{-\lambda^2}{4k}} \int_{-\infty}^{\infty} e^{-k\left(x - \frac{i\lambda}{2k}\right)^2} dx = (\alpha N)^{-2\alpha N} N^N e^{\frac{-\lambda^2}{4k}} \sqrt{\frac{\pi}{k}}$$

Let me verify now that things still work (see GVLongterm16.nb part 1)

Agreement is passable, but it's not immediately clear whether convergence is fast enough to ensure agreeing values in the limit. Let me try taking the integral over the logarithm. Upon doing this, (see GVLongterm16), I quickly reproduce the published result (Li, Park, Widom).

The next question is, where have all of the other terms gone? At what level of approximation did they disappear? Can I use the Bernoulli numbers to get higher-order terms? The Gaussian will certainly go to zero, so it appears that these correction terms may in fact be lost.

There were a few approximations to consider: I approximated the binomial entries using Stirling's approximation, I expanded the natural logarithm in its series, and then I used the Euler-MacLaurin formula to convert the sum to an integral. Somewhere, higher-order terms were lost. Let me try to deduce where this was:

Taking the Stirling approximation is fine as long as N is large. In the infinite-N limit here, this is entirely plausible for the numerator of the binomial, but I have taken also the

Stirling approximation of the denominator, admittedly reaching order-N and order-1 in the two terms near the upper-right and lower-left corners of the matrix. The order-1 term might be a source of error.

Next, I expanded $\left(1+\frac{xD}{\alpha N}\right)\ln\left(1+\frac{xD}{\alpha N}\right)+\left(1-\frac{xD}{\alpha N}\right)\ln\left(1-\frac{xD}{\alpha N}\right)$ and took only the lowest-order terms. This seems like a very likely source of loss of higher-order terms.

Finally, I used the Euler-MacLaurin formula to collapse the sum. Examining the correction terms here, however, it just doesn't seem very plausible that these could have any significant effects at all—at infinity, this function and all of its derivatives go to zero and to outclass the factorials in the denominator would be difficult despite the fact that the Bernoulli numbers become rather large.

Then, let me first try simply taking a few more terms from the expansion of the logarithm.

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In[68]:=
    Simplify[Series[(1 + x) Log[1 + x] + (1 - x) Log[1 - x], {x, 0, 8}]]
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Out[68]= $x^2 + \frac{x^4}{6} + \frac{x^6}{15} + \frac{x^8}{28} + O[x]^9$

This would leave me with the generating function

$$f_N(\lambda, D) = \sum_{-\infty}^{\infty} \binom{N}{\alpha N + xD} e^{i\lambda x} \approx (\alpha N)^{-2\alpha N} N^N \sum_{x=-\infty}^{\infty} e^{-\alpha N\left(\frac{xD}{\alpha N}\right)^2} e^{-\frac{\alpha N}{6}\left(\frac{xD}{\alpha N}\right)^4} e^{-\frac{\alpha N}{15}\left(\frac{xD}{\alpha N}\right)^6} e^{i\lambda x} \qquad k = \frac{D^2}{\alpha N}$$

Applying Euler-MacLaurin in this case, however, is a much harder integral.