Ben Sauerwine
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This week I will try to make an expansion for the formula

$$\int_{-\pi}^{\pi} \ln f_N(\lambda, D) d\lambda$$

$$with$$

$$f_N(\lambda, D) = t_{N,0} - 4(-1)^{D-1}\left[\sin(\pi(1+N-N\alpha))\frac{\Gamma(1+N)\Gamma(D-N+N\alpha)}{4\pi\Gamma(D+1+N\alpha)}\right]\times$$

$$\left(\begin{array}{l} e^{-i\lambda}\,{}_{D+1}F_D\left(\{1\}\cup\bigcup_{j=1}^{D}\left\{\frac{D+j-1}{D}-\frac{N}{D}+\frac{N\alpha}{D}\right\};\bigcup_{j=1}^{D}\left\{\frac{D+j}{D}+\frac{N\alpha}{D}\right\};(-1)^D e^{-i\lambda}\right)+ \\ e^{i\lambda}\,{}_{D+1}F_D\left(\{1\}\cup\bigcup_{j=1}^{D}\left\{\frac{D+j-1}{D}-\frac{N}{D}+\frac{N\alpha}{D}\right\};\bigcup_{j=1}^{D}\left\{\frac{D+j}{D}+\frac{N\alpha}{D}\right\};(-1)^D e^{i\lambda}\right) \end{array}\right)$$

$$t_{N,0} = \binom{N}{\alpha N}$$

NB: In past weeks, in some cases I mistakenly included a cosine of lambda term in this definition. The one you see here is correct. This always appeared correctly in Mathematica notebooks, but perhaps not on the progress reports.

Presumably, if I could determine an expansion for the function inside the integral, I could then integrate the terms of the expansion individually.

Mathematica will quickly handle the expansion for the portion inside the natural logarithm, and I will return to it with an explicit form later. For now, I want to know how to handle the logarithm itself. I notice the expansion:

$$\log(z) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(z-1)^k}{k} \qquad |z-1|<1$$

This indicates that if I could find an upper bound for U my function $f_N(\lambda, D) \le U$ then I could write

$$\log(z) = \log\left(\frac{Uz}{U}\right) = \log\left(\frac{z}{U}\right) + \log(U) = \log(U) + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}\left(\frac{z}{U}-1\right)^k}{k} \qquad |z-1|<1$$

Thus I see that while the coefficients on my expansion might get rather complicated, I could at least perform the integral and later look for identities on this expansion.

I have seen from plots in previous weeks that the maximum of this function occurs at $\lambda = 0$. Thus, I have unambiguously that $U = f_N(0, D)$.

Now I'll need the series for $f_N(\lambda, D)$ so that I may substitute this into the logarithm in order to obtain the parameters to substitute again into

$$\log(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(z-1)^k}{k} \quad |z-1| < 1.$$

Now I will symbolically find the expansion of $f_N(\lambda, D)$, as I would like at least a recursion to find the particular elements. This expansion should include only even powers of lambda, since the function is even about $\lambda = 0$.

$$f_N(\lambda, D) = t_{N,0} - 4(-1)^{D-1}\left[\sin(\pi(1+N-N\alpha))\frac{\Gamma(1+N)\Gamma(D-N+N\alpha)}{4\pi\Gamma(D+1+N\alpha)}\right] \times$$

$$\left(\begin{array}{l} e^{-i\lambda}\,{}_{D+1}F_D\left(\{1\} \cup \bigcup_{j=1}^{D}\left\{\frac{D+j-1}{D} - \frac{N}{D} + \frac{N\alpha}{D}\right\}; \bigcup_{j=1}^{D}\left\{\frac{D+j}{D} + \frac{N\alpha}{D}\right\}; (-1)^D e^{-i\lambda}\right) + \\ e^{i\lambda}\,{}_{D+1}F_D\left(\{1\} \cup \bigcup_{j=1}^{D}\left\{\frac{D+j-1}{D} - \frac{N}{D} + \frac{N\alpha}{D}\right\}; \bigcup_{j=1}^{D}\left\{\frac{D+j}{D} + \frac{N\alpha}{D}\right\}; (-1)^D e^{i\lambda}\right) \end{array} \right)$$

$$t_{N,0} = \binom{N}{\alpha N}$$

$$\beta(N, \alpha, D) = -4(-1)^{D-1}\left[\sin(\pi(1+N-N\alpha))\frac{\Gamma(1+N)\Gamma(D-N+N\alpha)}{4\pi\Gamma(D+1+N\alpha)}\right]$$

$$F_D^{(x)}(z) = x!\left(\prod_{j=1}^{D}\frac{\left(\frac{D+j-1}{D} - \frac{N}{D} + \frac{N\alpha}{D}\right)_x}{\left(\frac{D+j}{D} + \frac{N\alpha}{D}\right)_x}\right) \times$$

$${}_{D+1}F_D\left(\{1+x\} \cup \bigcup_{j=1}^{D}\left\{\frac{D+j-1}{D} - \frac{N}{D} + \frac{N\alpha}{D} + x\right\}; \bigcup_{j=1}^{D}\left\{\frac{D+j}{D} + \frac{N\alpha}{D} + x\right\}; z\right)$$

$$\frac{\partial}{\partial z}F_D^{(x)}(z) = F_D^{(x+1)}(z)$$

$$f_N(\lambda, D) = t_{N,0} + \beta(N, \alpha, D)\left(e^{-i\lambda}F_D^{(0)}\left((-1)^D e^{-i\lambda}\right) + e^{i\lambda}F_D^{(0)}\left((-1)^D e^{i\lambda}\right)\right)$$

So now I have heavily simplified the form of my result in such a way that expansion should be symbolically concise. Note that above, the Pochhammer symbol is used so that $(y)_x = \frac{\Gamma(y+x)}{\Gamma(y)}$. Thus, I have:

$$f_N(\lambda,D)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\frac{\partial^{2n}}{\partial\lambda^{2n}}\left[f_N(\lambda,D)\right]\Big|_{\lambda=0}\lambda^{2n}$$

$$\left[f_N(\lambda,D)\right]\Big|_{\lambda=0}=f_N(0,D)$$

$$\frac{\partial}{\partial\lambda}\left[f_N(\lambda,D)\right]=\beta(N,\alpha,D)\begin{bmatrix}-ie^{-i\lambda}F_D^{(0)}\left((-1)^De^{-i\lambda}\right)-e^{-i\lambda}\left(i(-1)^De^{-i\lambda}\right)F_D^{(1)}\left((-1)^De^{-i\lambda}\right)\\+ie^{i\lambda}F_D^{(0)}\left((-1)^De^{i\lambda}\right)+e^{i\lambda}\left(i(-1)^De^{-i\lambda}\right)F_D^{(1)}\left((-1)^De^{i\lambda}\right)\end{bmatrix}$$

$$=\beta(N,\alpha,D)\begin{bmatrix}-ie^{-i\lambda}F_D^{(0)}\left((-1)^De^{-i\lambda}\right)-ie^{-2i\lambda}(-1)^DF_D^{(1)}\left((-1)^De^{-i\lambda}\right)\\+ie^{i\lambda}F_D^{(0)}\left((-1)^De^{i\lambda}\right)+ie^{2i\lambda}(-1)^DF_D^{(1)}\left((-1)^De^{i\lambda}\right)\end{bmatrix}$$

$$\frac{\partial}{\partial\lambda}\beta(N,\alpha,D)\begin{bmatrix}-ie^{-i\lambda}F_D^{(0)}\left((-1)^De^{-i\lambda}\right)-ie^{-2i\lambda}(-1)^DF_D^{(1)}\left((-1)^De^{-i\lambda}\right)\\+ie^{i\lambda}F_D^{(0)}\left((-1)^De^{i\lambda}\right)+ie^{2i\lambda}(-1)^DF_D^{(1)}\left((-1)^De^{i\lambda}\right)\end{bmatrix}$$

$$=\beta(N,\alpha,D)\begin{bmatrix}-e^{-i\lambda}F_D^{(0)}\left((-1)^De^{-i\lambda}\right)-2e^{-2i\lambda}(-1)^DF_D^{(1)}\left((-1)^De^{-i\lambda}\right)\\-e^{i\lambda}F_D^{(0)}\left((-1)^De^{i\lambda}\right)-2e^{2i\lambda}(-1)^DF_D^{(1)}\left((-1)^De^{i\lambda}\right)+\\-e^{-2i\lambda}(-1)^DF_D^{(1)}\left((-1)^De^{-i\lambda}\right)-e^{-3i\lambda}F_D^{(2)}\left((-1)^De^{-i\lambda}\right)\\-e^{2i\lambda}(-1)^DF_D^{(1)}\left((-1)^De^{i\lambda}\right)-e^{3i\lambda}F_D^{(2)}\left((-1)^De^{i\lambda}\right)\end{bmatrix}$$

$$=-\beta(N,\alpha,D)\begin{bmatrix}e^{-i\lambda}F_D^{(0)}\left((-1)^De^{-i\lambda}\right)+e^{i\lambda}F_D^{(0)}\left((-1)^De^{i\lambda}\right)+\\3(-1)^D\left[e^{-2i\lambda}F_D^{(1)}\left((-1)^De^{-i\lambda}\right)+e^{2i\lambda}F_D^{(1)}\left((-1)^De^{i\lambda}\right)\right]+\\e^{-3i\lambda}F_D^{(2)}\left((-1)^De^{-i\lambda}\right)+e^{3i\lambda}F_D^{(2)}\left((-1)^De^{i\lambda}\right)\end{bmatrix}$$

Now I want to consider the function

$$G_D^{(x)}(\lambda)=e^{-(x+1)i\lambda}F_D^{(x)}\left((-1)^De^{-i\lambda}\right)+e^{(x+1)i\lambda}F_D^{(x)}\left((-1)^De^{i\lambda}\right)$$

$$\frac{\partial^2}{\partial\lambda^2}G_D^{(x)}(\lambda)=\frac{\partial}{\partial\lambda}\begin{bmatrix}-i(x+1)e^{-(x+1)i\lambda}F_D^{(x)}\left((-1)^De^{-i\lambda}\right)-ie^{-(x+2)i\lambda}(-1)^DF_D^{(x+1)}\left((-1)^De^{-i\lambda}\right)\\+i(x+1)e^{(x+1)i\lambda}F_D^{(0)}\left((-1)^De^{i\lambda}\right)+ie^{(x+2)i\lambda}(-1)^DF_D^{(x+1)}\left((-1)^De^{i\lambda}\right)\end{bmatrix}$$

$$=-\begin{bmatrix}(x+1)^2e^{-(x+1)i\lambda}F_D^{(x)}\left((-1)^De^{-i\lambda}\right)+(x+2)e^{-(x+2)i\lambda}(-1)^DF_D^{(x+1)}\left((-1)^De^{-i\lambda}\right)\\+(x+1)^2e^{(x+1)i\lambda}F_D^{(x)}\left((-1)^De^{i\lambda}\right)+(x+2)e^{(x+2)i\lambda}(-1)^DF_D^{(x+1)}\left((-1)^De^{i\lambda}\right)\\+(x+1)e^{-(x+2)i\lambda}(-1)^DF_D^{(x+1)}\left((-1)^De^{-i\lambda}\right)+e^{-(x+3)i\lambda}(-1)^{2D}F_D^{(x+2)}\left((-1)^De^{-i\lambda}\right)\\+(x+1)e^{(x+2)i\lambda}(-1)^DF_D^{(x+1)}\left((-1)^De^{i\lambda}\right)+e^{(x+3)i\lambda}(-1)^{2D}F_D^{(x+2)}\left((-1)^De^{i\lambda}\right)\end{bmatrix}$$

$$=-(x+1)^2G_D^{(x)}(\lambda)-(2x+3)(-1)^DG_D^{(x+1)}(\lambda)-(-1)^{2D}G_D^{(x+2)}(\lambda)$$

So now I have a recurrence relation with which I can find later coefficients in my generating function’s Taylor expansion. To recap, then, I have

$$f_N(\lambda, D) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{\partial^{2n}}{\partial \lambda^{2n}} \left[f_N(\lambda, D) \right]\Big|_{\lambda=0} \lambda^{2n}$$

$$\frac{\partial^{2n}}{\partial \lambda^{2n}} \left[f_N(\lambda, D) \right]\Big|_{\lambda=0} \Bigg|_{\lambda=0} = \beta(N, \alpha, D) \frac{\partial}{\partial \lambda^{2n}} \left[G_D^{(0)}(\lambda, D) \right] \Bigg|_{\lambda=0}$$

$$G_D^{(x)}(\lambda) = e^{-(x+1)i\lambda} F_D^{(x)}\left((-1)^D e^{-i\lambda} \right) + e^{(x+1)i\lambda} F_D^{(x)}\left((-1)^D e^{i\lambda} \right)$$

$$\frac{\partial^2}{\partial \lambda^2} G_D^{(x)}(\lambda) = -(x+1)^2 G_D^{(x)}(\lambda) - (2x+3)(-1)^D G_D^{(x+1)}(\lambda) - (-1)^{2D} G_D^{(x+2)}(\lambda)$$

where

$$F_D^{(x)}(z) = x! \left(\prod_{j=1}^{D} \frac{\left(\frac{D+j-1}{D} - \frac{N}{D} + \frac{N\alpha}{D} \right)_x}{\left(\frac{D+j}{D} + \frac{N\alpha}{D} \right)_x} \right) \times$$

$$ {}_{D+1}F_D \left(\{1+x\} \cup \bigcup_{j=1}^{D} \left\{ \frac{D+j-1}{D} - \frac{N}{D} + \frac{N\alpha}{D} + x \right\}; \bigcup_{j=1}^{D} \left\{ \frac{D+j}{D} + \frac{N\alpha}{D} + x \right\}; z \right)$$

This information is sufficient to use as a recursion for the highest-order terms in the expansion of $f_N(\lambda, D)$.

Now using my upper bound, I write

$$\ln\left(f_N(D,\lambda) \right) = \ln\left(f_N(D,0) \right) + \ln\left(\frac{f_N(D,\lambda)}{f_N(D,0)} \right) = \ln\left(f_N(D,0) \right) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \left(\frac{f_N(D,\lambda)}{f_N(D,0)} - 1 \right)^k}{k}$$

Simplifying this, I’ll have

$$f_N(\lambda, D) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} f_N^{\,x} \lambda^{2n}$$

$$\ln\left(f_N^{\,0} \right) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \left(\sum_{n=0}^{\infty} \frac{f_N^{\,n}}{f_N^{\,0}} - 1 \right)^k}{k} = \ln\left(f_N^{\,0} \right) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \left(\sum_{n=1}^{\infty} \frac{f_N^{\,n}}{f_N^{\,0}} \right)^k}{k}$$

$$= \ln\left(f_N^{\,0} \right) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left[\left(\left(\frac{f_N^{\,2}}{f_N^{\,0}} \right) \lambda^2 \right)^k + k \left(\left(\frac{f_N^{\,2}}{f_N^{\,0}} \right) \lambda^2 \right)^{k-1} \frac{f_N^{\,4}}{f_N^{\,0}} \lambda^4 \right]$$

And now it is clear to see that to take my integral from this is useless, since this series’ radius of convergence is 1 and my integral runs from $-\pi$ to π and will rapidly become dictated by the high-order lambda terms and certainly once I reach $\lambda = 1$ I will be dealing with entirely the coefficients, which here have no particular inclination to be small (as shown last week).

I had hoped to avoid actually expanding in D: expanding in individual Pochhammer symbols, of course, will give essentially gamma functions. However, this function also involves D in the indices of the hypergeometric function: this indicates that somehow any expansion in D would also have to include the growth in number of Pochhammer symbols. Indeed, getting an idea of what I'm dealing with in a function, I have, if you will, a gamma function of separate gamma functions not unlike

$$\prod_{j=1}^{D}\frac{\Gamma\left(\frac{1}{D}(D+j+c)\right)}{\Gamma\left(\frac{1}{D}(D+c)\right)}.$$

It's not entirely clear what sort of expansion in D is necessary to make this continuous. I might be able to get around using the Hypergeometric function entirely, however, if I was able to individually expand the binomials that contribute to producing this generating function in large D and then take only the relevant terms. Unfortunately, expanding in large D is not possible since the function $\binom{N}{\alpha N+\frac{1}{d}x}$ has an essential singularity in the large-d limit.

It seems that I've hit a lot of dead ends here: I can't perform the integral over the hypergeometric function directly, in Fourier expansion, directly as a series, or in polynomial expansion. Further, I can't expand the individual binomials in large D. I might be able to expand the hypergeometric functions in D, but unfortunately I do not know even how to represent functions like the one described above in a continuous manner. Indeed, I even had to bend my initial assumptions because the Gamma function becomes undefined at negative integers.

So what can I do?

- I could try to invert the infinite matrix, and find the determinant of the inverse.
- I could try to find the eigenvalues of the infinite matrix.
- I could find a differential equation in D with solutions like $\prod_{j=1}^{D}\frac{\Gamma\left(\frac{1}{D}(D+j+c)\right)}{\Gamma\left(\frac{1}{D}(D+c)\right)}$, use that equation to find a continuous function in D satisfying this, then use a series solution to that equation to produce an expansion in large D for my Hypergeometric function (assuming that the function does not have an essential singularity there). This may be easier to integrate than the Hypergeometric function itself.

There is little doubt that this method of finding the determinant will work—I have shown in previous weeks that it converges to the expected result and the published approximation and so indeed there must be some identity bridging between them.

What I have determined so far is:

- Any valid solution must consider every element of the Gessel-Viennot matrix in this region. None may be assumed to be small.
- There is a relatively simple formula to bridge between a discrete determinant of the Gessel-Viennot matrix to the integral form, and it is applicable in this case.
- Conventional means of integration and expansion are do not seem to be of use in this case. The functions have essential singularities and expansions at many points of interest.