The Rayleigh Instability
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Let's investigate the Rayleigh instability of a long cylinder of radius R.

Suppose the cylinder seeks to reduce its surface area while maintaining its volume. Now let the cylinder be perturbed over a single wavelength λ so that

$$R_{new}(x) = r + \delta \sin\left(\frac{2\pi}{\lambda} x\right), \text{ with } \delta << R$$

Now:

$$A_{old} = 2\pi R \lambda$$
$$V_{old} = \pi R^2 \lambda$$
$$V_{new} = \int_0^\lambda \pi \left(r + \delta \sin\left(\frac{2\pi}{\lambda} x\right)\right)^2 dx = \frac{1}{2}\pi\left(d^2 + 2r^2\right)\lambda$$

The constraint that I maintain the same volume means that:

$$A_{old} = 2\pi R \lambda$$
$$V_{old} = V_{new}$$
$$\pi R^2 \lambda = \frac{1}{2}\pi\left(d^2 + 2r^2\right)\lambda$$
$$\sqrt{R^2 - \frac{1}{2}d^2} = r$$

Now I need to find the new area, using the arclength:

$$A_{new} = \int_0^\lambda 2\pi R_{new}(x) ds = \int_0^\lambda 2\pi R_{new}(x)\sqrt{dx^2 + dr^2} = \int_0^\lambda 2\pi R_{new}(x)\sqrt{1 + \left(\frac{dR_{new}}{dx}\right)^2} dx$$
$$= \int_0^\lambda 2\pi \left(r + \delta \sin\left(\frac{2\pi}{\lambda} x\right)\right)\sqrt{1 + \left(\frac{2\pi\delta}{\lambda}\right)^2 \cos^2\left(\frac{2\pi}{\lambda} x\right)} dx$$

Expanding the square root,

$$\sqrt{1 + \left(\frac{2\pi\delta}{\lambda}\right)^2 \cos^2\left(\frac{2\pi}{\lambda} x\right)} = 1 + \frac{1}{2}\left(\frac{2\pi\delta}{\lambda}\right)^2 \cos^2\left(\frac{2\pi}{\lambda} x\right) + \ldots$$

Then,

$$A_{new} \approx \int_0^{\lambda} 2\pi\left(r+\delta\sin\left(\frac{2\pi}{\lambda}x\right)\right)\left[1+\frac{1}{2}\left(\frac{2\pi\delta}{\lambda}\right)^2\cos^2\left(\frac{2\pi}{\lambda}x\right)\right]dx$$

$$= \int_0^{\lambda} 2\pi\left(r+\delta\sin\left(\frac{2\pi}{\lambda}x\right)\right)\left[1+\frac{1}{2}\left(\frac{2\pi\delta}{\lambda}\right)^2\left(1-\sin^2\left(\frac{2\pi}{\lambda}x\right)\right)\right]dx$$

$$\approx \int_0^{\lambda} 2\pi r\left[1+\frac{1}{2}\left(\frac{2\pi\delta}{\lambda}\right)^2\left(1-\sin^2\left(\frac{2\pi}{\lambda}x\right)\right)\right]dx$$

In the last step, I have dropped terms that integrate to zero and taken the highest order in δ. Then,

$$A_{new} \approx \int_0^{\lambda} 2\pi r\left[1+\frac{1}{2}\left(\frac{2\pi\delta}{\lambda}\right)^2\left(1-\sin^2\left(\frac{2\pi}{\lambda}x\right)\right)\right]dx = 2\pi r\lambda\left(1+\left(\frac{\pi\delta}{\lambda}\right)^2\right)$$

Now I take:

$$r = \sqrt{R^2-\frac{1}{2}\delta^2} = R\sqrt{1-\frac{1}{2}\left(\frac{\delta}{R}\right)^2} \approx 1-\frac{1}{4}\left(\frac{\delta}{R}\right)^2+\ldots$$

Then:

$$A_{new} \approx 2\pi R\lambda\left(1-\frac{1}{4}\left(\frac{\delta}{R}\right)^2\right)\left(1+\left(\frac{\pi\delta}{\lambda}\right)^2\right) \approx A_{old}\left[1-\frac{1}{4}\left(\frac{\delta}{R}\right)^2+\left(\frac{\pi\delta}{\lambda}\right)^2\right]$$

$$= A_{old}\left[1+\pi^2\delta^2\left(\frac{1}{\lambda^2}-\frac{1}{(2\pi R)^2}\right)\right]$$

Now I see clearly that under this perturbation of wavelength λ, the surface area increases as volume stays constant if $\lambda < 2\pi R$. This indicates that for wavelengths with $\lambda < 2\pi R$, the cylinder is stable. However, if $\lambda > 2\pi R$, surface area is decreased at constant volume and then the cylinder is unstable with respect to perturbations of this wavelength.

This "Rayleigh Instability", then, is the reason for droplet formation in a long stream of fluid.