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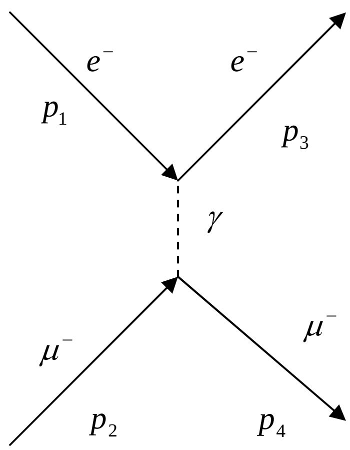
Nuclear and Particle Physics Homework 8

1) **Consider spinless $e^- \mu^- \to e^- \mu^-$ scattering. Calculate the cross section without neglecting either mass. Be careful not to use any expressions which themselves are only valid for negligible mass.**

The diagram for this scattering is:



And the Feynman rule say, then, that

$$-iM = \left(ie\left(p_1^\mu + p_3^\mu\right)\right)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\left(ie\left(p_2^\nu + p_4^\nu\right)\right)$$

Certainly, $p_1^2 = p_3^2 = m_e^2$ and $p_2^2 = p_4^2 = m_\mu^2$.

Further, I may use for convenience:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = m_e^2 + m_\mu^2 + 2p_1 \cdot p_2$$
$$= m_e^2 + m_\mu^2 + 2\left(E_e E_\mu + k_e k_\mu\right)$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 = 2m_\mu^2 - 2p_1 \cdot p_3 = 2m_e^2 - 2\left(E_e{}^2 - k_e{}^2 \cos\theta\right) = -2k_e{}^2(1 - \cos\theta)$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 = m_e^2 + m_\mu^2 - 2p_1 \cdot p_4 = m_\mu^2 + m_e^2 - 2\left(E_e E_\mu + k_e k_\mu \cos\theta\right)$$
$$= s - 4E_e E_\mu - 2k_e k_\mu \cos\theta$$

Incidentally, $q^2 = t$, the momentum transfer.

Expanding, then, I have:

$$-iM = \frac{ie^2}{t}\left(p_1 \cdot p_2 + p_1 \cdot p_4 + p_3 \cdot p_2 + p_3 \cdot p_4\right)$$

or in terms of the Mandelstam variables,

$$M = -e^2 \frac{s-u}{t}$$

Since these variables are Lorentz-invariant, I may now impose any frame I find convenient in order to find the relation for M . In this case, I may choose the center of mass frame where the momenta are the same: $p_1 = p_2 = p_3 = p_4 = k$.

$$M = -e^2 \frac{s-u}{t} = -e^2 \frac{2E_e E_\mu + k^2(1+\cos\theta)}{k^2(1-\cos\theta)}$$

Now

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{\left|\vec{p}_3^{\,*}\right|}{\left|\vec{p}_1^{\,*}\right|}\left|M_{fi}\right|^2 = \frac{e^4}{64\pi^2 s}\left|\frac{2E_e E_\mu + k^2(1+\cos\theta)}{k^2(1-\cos\theta)}\right|^2$$

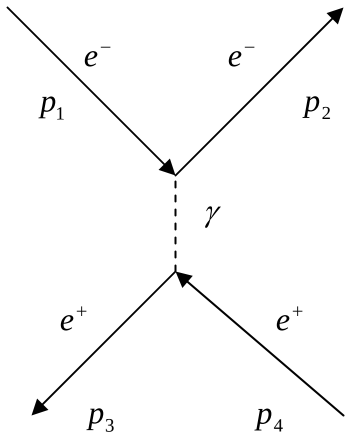
since in this CM frame, the magnitudes of the momenta are identical.

2) **Calculate the amplitude M for spinless $e^+e^- \to e^-e^+$ in terms of the Mandelstam variables, neglecting all masses.**

Now,

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = 2p_1 \cdot p_2$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 = -2p_1 \cdot p_3$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 = -2p_1 \cdot p_4$$

One diagram is:



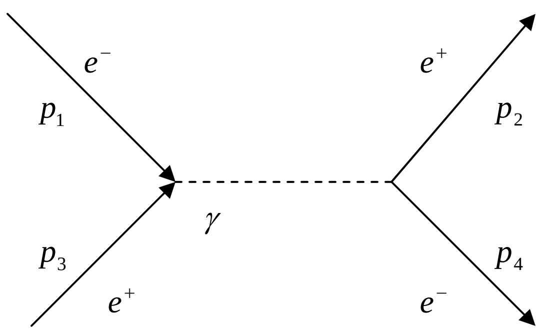
Here, $q^2 = t$, the momentum transferred in the photon line:

$$-iM_t = \left(ie\left(p_1^\mu + p_3^\mu\right)\right)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\left(ie\left(-p_2^\nu - p_4^\nu\right)\right)$$

$$= i\frac{e^2}{q^2}\left(-p_1 \cdot p_2 - p_1 \cdot p_4 - p_3 \cdot p_2 - p_3 \cdot p_4\right) = ie^2\frac{u-s}{t}$$

$$M_t = e^2\frac{s-u}{t}$$

The other is s-channel scattering:



Here, $q^2 = s$, the momentum transferred in the photon line:

$$-iM_s = \left(ie\left(-p_1^\mu + p_2^\mu\right)\right)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\left(ie\left(-p_3^\nu + p_4^\nu\right)\right)$$

$$= i\frac{e^2}{q^2}\left(p_1 \cdot p_3 - p_1 \cdot p_4 - p_2 \cdot p_3 + p_2 \cdot p_4\right) = ie^2\frac{u-t}{s}$$

$$M_s = e^2\frac{t-u}{s}$$

(just as in problem 1).

The sum is then:

$$M = e^2\left(\frac{s-u}{t} + \frac{t-u}{s}\right)$$

3) **The matrix element for $e^+e^- \to e^-e^+$ including the electron spin is**
$$|M|^2 = 2e^4\left(\frac{s^2+u^2}{t^2} + \frac{2u^2}{ts} + \frac{t^2+u^2}{s^2}\right)$$
Starting with this, calculate $\frac{d\sigma}{d\Omega}$ in terms of s and $\cos\theta$. You may neglect the electron mass.

Using the identities:

$$s = 4k^2$$

$$t = -2k^2(1-\cos\theta) = -\frac{1}{2}s(1-\cos\theta)$$

$$u = -2k^2(1+\cos\theta) = -\frac{1}{2}s(1+\cos\theta)$$

I will brutalize this with Mathematica's FullSimplify function:

$$|M|^2 = 2e^4\left(\frac{s^2+u^2}{t^2} + \frac{2u^2}{ts} + \frac{t^2+u^2}{s^2}\right) = \frac{e^4\left(7+\cos(2\theta)\right)^2}{4\left(\cos(\theta)-1\right)^2}$$

Then

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{\left|\vec{p}_3^{\,*}\right|}{\left|\vec{p}_1^{\,*}\right|}\left|M_{fi}\right|^2 = \frac{1}{64\pi^2 s}\frac{e^4\left(7+\cos(2\theta)\right)^2}{4\left(\cos(\theta)-1\right)^2}$$

since in this CM frame, the magnitudes of the momenta are identical.