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Nuclear and Particle Physics Homework 7

1) **Use isospin symmetry to write each of the three $I = 1$ ρ states as a sum of combinations of two $I = 1$ π states. How does each ρ decay?**

Coupling two $I = 1$ π states, I see that:

$$\left| j \quad m \right\rangle$$
$$\pi^{+} = \left|1 \quad 1\right\rangle \quad \pi^{0} = \left|1 \quad 0\right\rangle \quad \pi^{-1} = \left|1 \quad -1\right\rangle$$

I then seek the coefficients:

$$\rho^{+} = \left|1 \quad 1\right\rangle$$
$$\left\langle 1 \quad 1; \pi^{i} \quad \pi^{j} \middle| 1 \quad 1 \right\rangle$$
$$\left\langle 1 \quad 1; \pi^{+} \to 1 \quad \pi^{0} \to 0 \middle| 1 \quad 1 \right\rangle = \sqrt{\frac{1}{2}}$$
$$\left\langle 1 \quad 1; \pi^{0} \to 0 \quad \pi^{+} \to 1 \middle| 1 \quad 1 \right\rangle = -\sqrt{\frac{1}{2}}$$
$$\therefore \left|\rho^{+}\right\rangle = \sqrt{\frac{1}{2}}\left|\pi^{+}\pi^{0}\right\rangle - \sqrt{\frac{1}{2}}\left|\pi^{0}\pi^{+}\right\rangle$$

For the neutral ρ,

$$\rho^{0} = \left|1 \quad 0\right\rangle$$
$$\left\langle 1 \quad 1; \pi^{i} \quad \pi^{j} \middle| 1 \quad 0 \right\rangle$$
$$\left\langle 1 \quad 1; \pi^{+} \to 1 \quad \pi^{-} \to -1 \middle| 1 \quad 0 \right\rangle = \sqrt{\frac{1}{2}}$$
$$\left\langle 1 \quad 1; \pi^{-} \to -1 \quad \pi^{+} \to 1 \middle| 1 \quad 0 \right\rangle = -\sqrt{\frac{1}{2}}$$
$$\therefore \left|\rho^{0}\right\rangle = \sqrt{\frac{1}{2}}\left|\pi^{+}\pi^{-}\right\rangle - \sqrt{\frac{1}{2}}\left|\pi^{-}\pi^{+}\right\rangle$$

And for the negative ρ,

$\rho^- = |1 \quad -1\rangle$

$\langle 1 \quad 1; \pi^i \quad \pi^j | 1 \quad -1\rangle$

$\langle 1 \quad 1; \pi^0 \to 0 \quad \pi^- \to -1 | 1 \quad -1\rangle = -\sqrt{\frac{1}{2}}$

$\langle 1 \quad 1; \pi^- \to -1 \quad \pi^0 \to 0 | 1 \quad -1\rangle = \sqrt{\frac{1}{2}}$

$\therefore |\rho^-\rangle = \sqrt{\frac{1}{2}}|\pi^-\pi^0\rangle - \sqrt{\frac{1}{2}}|\pi^0\pi^-\rangle$

In all cases, I have omitted the Clebsch-Gordan coefficients which are zero. I have seen that each ρ decays to a pair of π as listed above.

2) Consider a world with only $u, \bar{u}, d, \bar{d}$ quarks. For any state, let N_u denote the number of u quarks, $N_{\bar{u}}$ denote the number of $\bar{u}$ quarks, and similarly for $d, \bar{d}$.

a) Write down separate equations for the charge Q, baryon number B, and isospin projection I_3, in terms of $N_u, N_{\bar{u}}, N_d, N_{\bar{d}}$ only.

$$Q = \frac{2}{3}(N_u - N_{\bar{u}}) + \frac{1}{3}(N_{\bar{d}} - N_d)$$

$$B = \frac{1}{3}(N_u + N_d) - \frac{1}{3}(N_{\bar{u}} + N_{\bar{d}})$$

$$I_3 = \frac{1}{2}(N_u + N_{\bar{d}}) - \frac{1}{2}(N_d + N_{\bar{u}})$$

b) Consider new quantum numbers:
-Upness: $U \equiv N_u - N_{\bar{u}}$
-Downness: $D \equiv N_d - N_{\bar{d}}$
Solve for each of these U, D in three different ways:
-In terms of B and I_3 only.
-In terms of Q and I_3 only.
-In terms of Q and B only.

$$Q = \frac{2}{3}\left(N_u - N_{\bar{u}}\right) + \frac{1}{3}\left(N_{\bar{d}} - N_d\right) = \frac{2}{3}U - \frac{1}{3}D$$
$$B = \frac{1}{3}\left(N_u + N_d\right) - \frac{1}{3}\left(N_{\bar{u}} + N_{\bar{d}}\right) = \frac{1}{3}U + \frac{1}{3}D$$
$$I_3 = \frac{1}{2}\left(N_u + N_{\bar{d}}\right) - \frac{1}{2}\left(N_d + N_{\bar{u}}\right) = \frac{1}{2}U - \frac{1}{2}D$$

Then in terms of B and I_3,

$$B = \frac{1}{3}U + \frac{1}{3}D \quad I_3 = \frac{1}{2}U - \frac{1}{2}D$$

$\therefore$

$$U = \frac{3}{2}B + I_3 \quad D = \frac{3}{2}B - I_3$$

In terms of Q and I_3:

$$Q = \frac{2}{3}U - \frac{1}{3}D \quad I_3 = \frac{1}{2}U - \frac{1}{2}D$$

$\therefore$

$$U = -2I_3 + 3Q \quad D = -4I_3 + 3Q$$

In terms of Q and B:

$$Q = \frac{2}{3}U - \frac{1}{3}D \quad B = \frac{1}{3}U + \frac{1}{3}D$$

$\therefore$

$$U = B + Q \quad D = 2B - Q$$

3) **Let's examine how problem 2 is modified if we also consider s and $\bar{s}$ quarks along with a new quantum number, $S = -\left(N_s - N_{\bar{s}}\right)$ (the – sign is a historical convention).**

a) **Write equations for Q, B, I_3 as above, now including the number of s and $\bar{s}$ quarks.**

$$Q = \frac{2}{3}\left(N_u - N_{\bar{u}}\right) + \frac{1}{3}\left(N_{\bar{d}} - N_d\right) + \frac{1}{3}\left(N_{\bar{s}} - N_s\right) = \frac{2}{3}U - \frac{1}{3}D + \frac{1}{3}S$$
$$B = \frac{1}{3}\left(N_u + N_d + N_s\right) - \frac{1}{3}\left(N_{\bar{u}} + N_{\bar{d}} + N_{\bar{s}}\right) = \frac{1}{3}U + \frac{1}{3}D - \frac{1}{3}S$$
$$I_3 = \frac{1}{2}\left(N_u + N_{\bar{d}}\right) - \frac{1}{2}\left(N_d + N_{\bar{u}}\right) = \frac{1}{2}U - \frac{1}{2}D$$

b) Are U, D still redundant? Specifically, count the number of "N" variables, the number of constraints, etc., and explain what happens when the new s quark is added into the mix.

I now have the same number quantum numbers S, U, D as constraints Q, B, I_3, so there is no redundancy in my information. Specifically, solving this set I have:

$$U = B + Q$$
$$D = B - 2I_3 + Q$$
$$S = -B - 2I_3 + 2Q$$

4) Look over the baryon wave function and magnetic moment section of Povh Chapter 15.

a) Construct the properly symmetrized wave functions for $|\Lambda^0 \uparrow\rangle$ and $|\Sigma^0 \uparrow\rangle$. You may use without proof the statement that for Σ^0, the u and d quarks couple to give both spin and isospin 1, while for Λ^0, both of these are zero.

The difference between Λ^0 and Σ^0 is that the Λ^0 particle has isospin 0 and the Σ^0 has isospin 1.

First consider the particle $|\Lambda^0 \uparrow\rangle$. The first property is that isospin is zero, indicating that the up- and down- portions have :

$$|\Lambda^0{}_{ud}\rangle = \frac{1}{\sqrt{2}}\left(|u \uparrow d \downarrow\rangle - |u \downarrow d \uparrow\rangle\right)$$

this leaves that the strange portion must be $|\Lambda_s^0\rangle = |s \uparrow\rangle$. Now:

$$|\Lambda^0 \uparrow\rangle = \frac{1}{\sqrt{2}}\left(|u \uparrow d \downarrow s \uparrow\rangle - |u \downarrow d \uparrow s \uparrow\rangle\right)$$

However, due to symmetry between exchange in up and down quark, I have

$$|\Lambda^0 \uparrow\rangle = \frac{1}{2}\left(|u \uparrow d \downarrow s \uparrow\rangle - |u \downarrow d \uparrow s \uparrow\rangle - |d \uparrow u \downarrow s \uparrow\rangle + |d \downarrow u \uparrow s \uparrow\rangle\right)$$

Now consider the particle $|\Sigma^0 \uparrow\rangle$. Since the up-down and strange portions must now couple to spin one, I see from the Clebsch-Gordan coefficients that:

$$\left|S = \frac{1}{2}, S_z = \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|S_{ud} = 1, S_s = -\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|S_{ud} = 0, S_s = \frac{1}{2}\right\rangle$$

Then, applying the symmetry of exchange between up and down and leaving factors for clarity,

$$\left|\Sigma^0 \uparrow\right\rangle = \sqrt{\frac{2}{3}} \cdot \sqrt{\frac{1}{2}}\left(\left|u \uparrow d \uparrow s \downarrow\right\rangle + \left|d \uparrow u \uparrow s \downarrow\right\rangle\right)$$

$$-\sqrt{\frac{1}{3}} \cdot \frac{1}{2}\left(\left|u \uparrow d \downarrow s \uparrow\right\rangle - \left|u \downarrow d \uparrow s \uparrow\right\rangle - \left|d \uparrow u \downarrow s \uparrow\right\rangle + \left|d \downarrow u \uparrow s \uparrow\right\rangle\right)$$

b) Use these to calculate the quark model prediction for the magnetic moments: $\left\langle\Lambda^0\right|\mu\left|\Lambda^0\right\rangle, \left\langle\Sigma^0\right|\mu\left|\Sigma^0\right\rangle, \left\langle\Lambda^0\right|\mu\left|\Sigma^0\right\rangle$**.**

Simply adding up the moment corresponding to each individual member ket separately I get:

$$\left\langle\Lambda^0 \uparrow\right|\mu\left|\Lambda^0 \uparrow\right\rangle = \frac{1}{2}\left(\mu_u - \mu_d + \mu_s\right) + \frac{1}{2}\left(-\mu_u + \mu_d + \mu_s\right) = \mu_s$$

$$\left\langle\Sigma^0 \uparrow\right|\mu\left|\Sigma^0 \uparrow\right\rangle = \frac{2}{3}\mu_\mu + \frac{2}{3}\mu_d + \frac{2}{3}\mu_s - \frac{4}{12}\mu_s = \frac{2}{3}\mu_\mu + \frac{2}{3}\mu_d + \frac{1}{3}\mu_s$$

Finally, I see that my functions $\left|\Lambda^0 \uparrow\right\rangle$ and $\left|\Sigma^0 \uparrow\right\rangle$ have overlapping kets:

$$\left|\Lambda^0 \uparrow\right\rangle_{relevant} = \frac{1}{2}\left(\left|u \uparrow d \downarrow s \uparrow\right\rangle - \left|u \downarrow d \uparrow s \uparrow\right\rangle - \left|d \uparrow u \downarrow s \uparrow\right\rangle + \left|d \downarrow u \uparrow s \uparrow\right\rangle\right)$$

$$\left|\Sigma^0 \uparrow\right\rangle_{relevant} = -\sqrt{\frac{1}{3}} \cdot \frac{1}{2}\left(\left|u \uparrow d \downarrow s \uparrow\right\rangle - \left|u \downarrow d \uparrow s \uparrow\right\rangle - \left|d \uparrow u \downarrow s \uparrow\right\rangle + \left|d \downarrow u \uparrow s \uparrow\right\rangle\right)$$

Again taking matching kets, multiplying, and adding the contributions, I see that

$$\left\langle\Lambda^0 \uparrow\right|\mu\left|\Sigma^0 \uparrow\right\rangle = \frac{1}{\sqrt{3}}\left(\mu_d - \mu_u\right)$$