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Nuclear and Particle Physics Homework 5

1) **Consider the kinematics of $e^+e^- \to \mu^+\mu^-$ scattering in the center-of-mass. The electrons have momentum k and the muons emerge at an angle θ relative to the electron and positron direction. Evaluate the expressions for s, t and u, retaining both m_e and m_μ.**

Suppose that these beams are colliding head-on. With p indicating the four-momenta,

$$p_{e-} = \begin{bmatrix} \sqrt{m_e^2 + k^2} \\ k \\ 0 \\ 0 \end{bmatrix} \quad p_{e+} = \begin{bmatrix} \sqrt{m_e^2 + k^2} \\ -k \\ 0 \\ 0 \end{bmatrix} \quad p_{\mu-} = \begin{bmatrix} \sqrt{m_\mu^2 + k'^2} \\ k'\cos\theta \\ k'\sin\theta \\ 0 \end{bmatrix} \quad p_{\mu+} = \begin{bmatrix} \sqrt{m_\mu^2 + k'^2} \\ -k'\cos\theta \\ -k'\sin\theta \\ 0 \end{bmatrix}$$

$Define:$

$$k'^2 \equiv m_e^2 + k^2 - m_\mu^2$$

$$s \equiv (p_{e+} + p_{e-})^2 = (p_{\mu+} + p_{\mu-})^2$$
$$(p_{e+} + p_{e-})^2 = 2m_e^2 + 2p_{e+} \cdot p_{e-} = 2m_e^2 + 2\left(m_e{}^2 + k^2 + k^2\right)$$
$$= 4m_e^2 + 4k^2$$

$$t \equiv (p_{e+} - p_{\mu+})^2 = (p_{e-} - p_{\mu-})^2$$
$$(p_{e+} - p_{\mu+})^2 = m_e^2 + m_\mu^2 - 2p_{e+}p_{\mu+}$$
$$= m_e^2 + m_\mu^2 - 2\left(\sqrt{m_e^2 + k^2}\right)\left(\sqrt{m_\mu^2 + k'^2}\right) + 2(kk'\cos\theta)$$
$$= m_e^2 + m_\mu^2 - 2\left(m_e^2 + k^2\right) + 2(kk'\cos\theta)$$
$$= m_\mu^2 - m_e^2 - 2k^2 + 2(kk'\cos\theta)$$

Simplification was possible since there is no reason to believe the muon energies should differ from the electron energies in this frame.

k’ is defined above.

$$u \equiv (p_{e+} - p_{\mu-})^2$$
$$= m_e^2 + m_\mu^2 - 2p_{e+}p_{\mu-} = m_e^2 + m_\mu^2 - 2\left(\sqrt{m_e^2 + k^2}\right)\left(\sqrt{m_\mu^2 + k'^2}\right) - 2(kk'\cos\theta)$$
$$= m_\mu^2 - m_e^2 - 2k^2 - 2(kk'\cos\theta)$$

k’ is defined above.

2) **This problem concerns fixed target collisions of protons or anti-proton beams with a proton target.**

a) **What is the expression for s in terms of the beam proton energy E and the mass m_p?**

$$s = \left(\begin{bmatrix} E \\ \sqrt{E^2 - m_p^2} \end{bmatrix} + \begin{bmatrix} m_p \\ 0 \end{bmatrix} \right)^2 = 2m_p^2 + 2Em_p$$

b) **What is the heaviest new particle that can be pair-produced (i.e., $pp \to ppX\overline{X}$) in the collision of an 800 GeV proton with a proton at rest? (Don't forget the protons in the final state!)**

At the threshold, $\sqrt{s}$ is the available energy in the CM frame:

$$\sqrt{s} = \sqrt{2m_p^2 + 2 \cdot 800GeV \cdot m_p} = 2m_p + 2m_X = \sqrt{2(938.28)^2 + 2 \cdot (800000) \cdot (938.28)} = 38769MeV$$

$$\frac{38769MeV - 2m_p}{2} = m_X = 18446MeV$$

Looking at a list of particles available from Wikipedia, it looks like one could well get a bottom quark from this interaction. It doesn't look like one could pair-produce any of the bosons predicted by the standard model. It seems unfortunate that despite all of this energy, only a small portion is available in the resulting CM frame for pair production. If they wanted something bigger, they should try a heavier target like some Lead.

c) **Repeat b but with $p\overline{p} \to X\overline{X}$. Does this answer make sense?**

$$\sqrt{s} = 2m_X$$

$$m_X = \frac{38769MeV}{2} = 19385MeV$$

This makes perfect sense—there is more available energy in the CM frame.

Again, it looks like the heaviest particle possible is a bottom quark.

3) **An electron of energy $E = 4.88GeV$ scatters at $\theta = 10^\circ$ from a proton of mass M at rest. The scattered electron has energy E' and the proton is inelastically excited to a state of invariant mass W.**

a) **Calculate E' in terms of E, W, M, θ.**

Let m_e be negligibly small compared to E.

Then:

$$\left(\begin{bmatrix} E \\ E \\ 0 \end{bmatrix} - \begin{bmatrix} E' \\ E'\cos\theta \\ E'\sin\theta \end{bmatrix}\right)^2 = \left(\begin{bmatrix} M \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} E_N \\ E - E'\cos\theta \\ -E'\sin\theta \end{bmatrix}\right)^2$$

$$2{m_e}^2 - 2EE'(1-\cos\theta) = M^2 + W^2 - 2ME_N$$

$$\frac{2{m_e}^2}{2M} - E'\frac{E}{M}(1-\cos\theta) = \frac{M^2+W^2}{2M} - E_N$$

$$\frac{m_e}{M} \approx 0$$

$$\therefore E'\frac{E}{M}(1-\cos\theta) = E_N - \frac{M^2+W^2}{2M}$$

$$Energy \quad conservation: \quad M + E = E_N + E'$$

$$E'\frac{E}{M}(1-\cos\theta) = M + E - E' - \frac{M^2+W^2}{2M}$$

$$E'\left(1 + \frac{E}{M}(1-\cos\theta)\right) = M + E - \frac{M^2+W^2}{2M}$$

$$E' = \frac{E + \dfrac{M^2-W^2}{2M}}{1 + \dfrac{E}{M}(1-\cos\theta)}$$

b) What is the answer when $W = M$? Does this make sense?

$$E' = \frac{E + \dfrac{M^2-W^2}{2M}}{1 + \dfrac{E}{M}(1-\cos\theta)} \rightarrow \frac{E}{1 + \dfrac{E}{M}(1-\cos\theta)}.$$

This is just the Compton scattering formula—for elastic scattering (whereas the nucleus has not picked up any bonus mass).

c) Use this formula to calculate E' for the cases of $W = M$, $W = 1230MeV$, and $W = 1520MeV$.

$$E'\left(W=M\right)=\frac{4880MeV}{1+\frac{4880MeV}{938.3MeV}\left(1-\cos 10^{\circ}\right)}=4523MeV$$

$$E'\left(W=1230MeV\right)=\frac{4880MeV+\frac{\left(938.3MeV\right)^2-\left(1230MeV\right)^2}{2\cdot 938.3MeV}}{1+\frac{4880MeV}{938.3MeV}\left(1-\cos 10^{\circ}\right)}=4211MeV$$

$$E'\left(W=1520MeV\right)=\frac{4880MeV+\frac{\left(938.3MeV\right)^2-\left(1520MeV\right)^2}{2\cdot 938.3MeV}}{1+\frac{4880MeV}{938.3MeV}\left(1-\cos 10^{\circ}\right)}=3816MeV$$

d) Consider the limit from part (a) where $W-M<<M$. Explain how this limit allows us to easily obtain energy levels of the Oxygen nucleus from the data in Povh 6.3.

$$E'=\frac{E+\frac{M^2-W^2}{2M}}{1+\frac{E}{M}\left(1-\cos\theta\right)}\rightarrow\frac{E+\frac{\left(M+W\right)\left(M-W\right)}{2M}}{1+\frac{E}{M}\left(1-\cos\theta\right)}$$

$$\alpha=W-M$$

$$\frac{\left(M+W\right)\left(M-W\right)}{2M}\rightarrow-\frac{2M\alpha+\alpha^2}{2M}\approx-\alpha$$

$$\therefore E'_{W-M<<M}\approx\frac{E-\left(W-M\right)}{1+\frac{E}{M}\left(1-\cos\theta\right)}=$$

$$W_{W-M<<M}=E+M-E'\left[1+\frac{E}{M}\left(1-\cos\theta\right)\right]$$

From Povh 6.3, it is given that $E=246MeV,\theta=148.5^{\circ}$. The rest mass of an Oxygen-16 nucleus is $\approx 8M_H+8M_n=8\cdot 938.78MeV+8\cdot 939.57MeV=15027MeV$.

Of course, this approximation seems unnecessary: Instead, I can take the equally benign and more accurate formula from the original expression

$$W^2=M^2-2M\left(\left[1+\frac{E}{M}\left(1-\cos\theta\right)\right]E'-E\right)$$

All that's left, then, is to read off the E' at the spikes to find the excited states. The big one, for example, corresponds to roughly $E=160$ for

$W_1 \approx 15108 MeV$.

4) **When thinking about the number of independent parameters in deep inelastic scattering, we decided that there were two independent variables which we chose as q^2 and W. That means that any other scalar quantities can be expressed in terms of these. Taking the electron 4-momentum before (after) scattering to be $p(p')$, with $q = p - p'$, and the 'target system' before (after) to be $P(P')$, express the following in terms of q^2, W, and the proton mass M. We continue to neglect m_e.**

- $P \cdot q$
- $P' \cdot q$
- $p \cdot p'$

(fill in derivation of first one).

$$W^2 = (P')^2 = (P+q)^2 = M^2 + q^2 + 2P \cdot q$$
$$\therefore P \cdot q = \frac{W^2 - M^2 - q^2}{2}$$

Next,

$$p + P = p' + P'$$
$$P + p - p' = P'$$
$$P + q = P'$$
$$(P+q) \cdot q = P' \cdot q$$
$$\therefore P' \cdot q = P \cdot q + q^2 = \frac{W^2 - M^2 + q^2}{2}$$

Finally,

$$-\frac{1}{2}q^2 = -(p-p') \cdot (p-p') = -\frac{1}{2}\left(p^2 + p'^2 - 2p \cdot p'\right) = -\frac{1}{2}m_e^2 + p \cdot p' \approx p \cdot p'$$