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Nuclear and Particle Physics Homework 4

1) **Consider elastic scattering of relativistic electrons from a nucleus.**

a) **Write down the formula for $\frac{E_e'}{E_e}$, the ratio of electron energies after and before the scatter. You may neglect the electron mass, but not the recoil energy. Use the formula to find two conditions on the electron's kinematics that would cause the recoil energy of the nucleus being negligible.**

The Compton scattering equation (usually used for photons, but a very high energy electron mimics a photon in that its mass is negligible).

$$\frac{E_\gamma'}{E_\gamma} = \frac{1}{1 + \frac{E_\gamma}{m}(1 - \cos\theta)}$$

Thus, either $\frac{E_\gamma}{m}$ is small or θ is small in order to make recoil energy negligible.

b) **If the nuclear recoil energy is negligible, then we can take $|\vec{p}_e| = |\vec{p}_e'|$. In this case, find the direction of the momentum transfer $\vec{q} \equiv \vec{p}_e - \vec{p}_e'$ with respect to the original beam direction in terms of the scattering angle θ of the electron.**

In this case, then, if the recoil energy is negligible then the vectors $\vec{p}_e, \vec{p}_e'$ form the legs of an isosceles triangle, with angle between them θ. Then the angle between $\vec{p}_e, \vec{q}$ is the external angle $\pi - \frac{\pi - \theta}{2} = \frac{\pi + \theta}{2}$.

2) **Derive the expression for the form-factor $F(q^2)$ for scattering from a homogeneous sphere of charge by evaluating the integral:**
$\int d^3 r e^{i\vec{q}\cdot\vec{r}} f(\vec{r})$.
As your first step, you may the result that, for spherically symmetric distributions,
$$F(q) = 4\pi \int r^2 dr \frac{\sin(qr)}{qr} f(r).$$

For a particular distribution, $f(r) = N\theta(R - r)$

Normalization (to the total amount of charge): $N = \frac{1}{\frac{4}{3}\pi R^3}$.

$$F(q) = \frac{4\pi N}{q}\int_0^R dr r \sin(qr)$$

$$\begin{aligned} u &= r & dv &= \sin(qr)dr \\ du &= dr & v &= -\frac{1}{q}\cos(qr) \end{aligned}$$

$$F(q) = \frac{4\pi N}{q}\left(-r\frac{1}{q}\cos(qr)\Big|_0^R + \int_0^R dr\frac{1}{q}\cos(qr)\right)$$

$$= \frac{4\pi N}{q}\left(-\frac{R}{q}\cos(qR) + \frac{1}{q^2}(\sin(qR) - \sin(0))\right) = \frac{3}{qR^3}\left(-\frac{R}{q}\cos(qR) + \frac{1}{q^2}\sin(qR)\right)$$

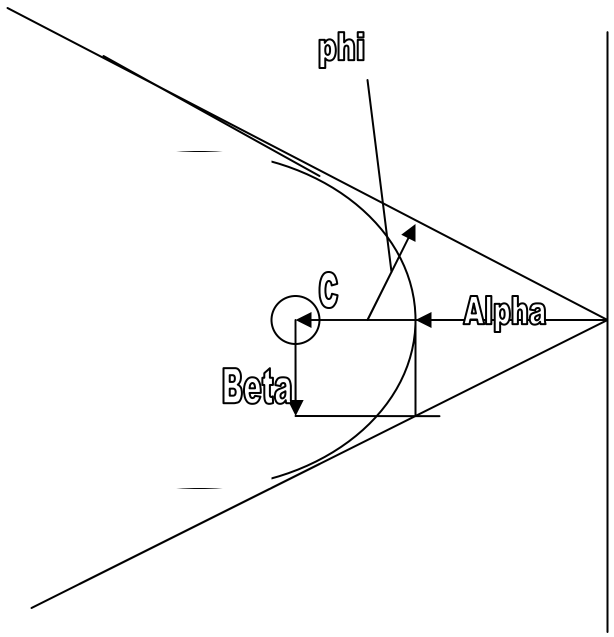
$$= \frac{3}{(qR)^3}(-qR\cos(qR) + \sin(qR))$$

3) **Consider the Rutherford scattering of an electron of energy E from a nucleus of charge Z (which may be treated as infinitely heavy). Find a relation between the impact parameter, b, of the final trajectory and the deflection, θ, of the electron.**
You will need the EM potential, two well-known conservation laws, and some analytic geometry of a conic section.
You may take as a given that orbits are given by conic sections.

Results will have two asymptotes—and so be hyperbolic.

$$V = -k\frac{eZ}{\sqrt{x^2 + y^2}}$$

The pure Lagrangian is rather difficult equations to solve; instead, consider a hyperbola in standard position, so that I may construct the appropriate impact parameter. If I take the center to be at the origin, then I have the general equation:



$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$

This equation has foci at:

$$c = \sqrt{\alpha^2 + \beta^2}$$

Further, I get twin asymptotes along the lines:

$$y = \pm \frac{\beta}{\alpha} x$$

Geometrically speaking, then, the impact parameter will correspond to the perpendicular distance between the focus and this line; namely,

$$b = \left| \frac{\begin{bmatrix} -\beta \\ \alpha \end{bmatrix} \begin{bmatrix} c \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|} \right| = \beta \;\; \textbf{(1)}$$

Next I am going to need to invoke conservation of angular momentum (note that I don't yet know where my focus lies: once I can eliminate both alpha and beta, I'll be ready to solve.)

I can relate conserved angular momentum to find the distance of closest approach:

$$L = mvr = C = m\left[\sqrt{\frac{2}{m}\left(E + k\frac{eZ}{c-\alpha}\right)}\right](c-\alpha) = m\left[\sqrt{\frac{2}{m}\left(E + k\frac{eZ}{\sqrt{\alpha^2+\beta^2}-\alpha}\right)}\right]\left(\sqrt{\alpha^2+\beta^2}-\alpha\right) \textbf{ (2)}$$

Now I want to find the angular momentum in the long-term limit:

$$L = m\left|v \times r\right| = m\sqrt{\frac{2}{m}}Eb = \sqrt{2mE}b \textbf{ (3)}$$

Now all that's left is to use equation (3) to eliminate (L) from equation 2 and solve (2) and (1) simultaneously for alpha:

$$\sqrt{2mE}b = m\left[\sqrt{\frac{2}{m}\left(E + k\frac{eZ}{\sqrt{\alpha^2+b^2}-\alpha}\right)}\right]\left(\sqrt{\alpha^2+b^2}-\alpha\right)$$

$$\alpha = \frac{keZ}{2E}$$

Note that the final scattering angle will be $\theta = \pi - 2\phi$, where $\tan\phi = \frac{\beta}{\alpha}$ (see diagram)

$$\theta = \pi - 2\arctan\left(\frac{2Eb}{keZ}\right)$$

$$b = \frac{keZ}{2E}\cot\left(\frac{\theta}{2}\right)$$

4) **We know that** $\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left|F(q^2)\right|^2$ **.**

Let's approximate the ^{40}Ca **nucleus as a homogeneous sphere with a sharp edge. Then, the first diffraction minimum corresponds to** $qR \approx 4.5$ **.**

Obtain an approximate value of R from Povh fig. 5.8 and then calculate the approximate electron beam energy used to take the data in Povh fig 5.7.

From Povh 5.8, I see that R is approximately 3 fm.

Then, $q = \frac{3}{2}\,fm^{-1}$. Further, I notice a that the first minimum is at approximately $\theta = 21^\circ$.

$$q = \frac{3}{2}\,fm^{-1}(\hbar c) = \frac{3}{2}\,fm^{-1}(197MeV \cdot fm) = 296MeV$$

From Povh equation (5.37), I see that the momentum transfer and momentum of the incident beam can be related by:

$$\lambda = \frac{\hbar}{|\vec{q}|} = \frac{\hbar}{|\vec{p}|}\frac{1}{2\sin\frac{\theta}{2}}$$

$$|\vec{p}| = \frac{|\vec{q}|}{2\sin\frac{\theta}{2}} = 946MeV$$

5) **This is a continuation of problem 4. Still assuming a homogeneous sphere, calculate the differential cross section (at the energy you found in #4) scattering off ^{40}Ca in 5 degree increments of angle from 25 to 60 degrees.**

Assuming extremely relativistic electrons, I will take the approximation where

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{4Z^2\alpha^2E'^2}{q^4}\cos^2\frac{\theta}{2}$$

$$E' \approx k = 946MeV$$

$$q = 2k\sin\frac{\theta}{2}$$

$$Z = 20$$

$$\alpha \approx \frac{1}{137}$$

$$\frac{4Z^2\alpha^2E'^2}{16k^4}\frac{\cos^2\frac{\theta}{2}}{\sin^4\frac{\theta}{2}} \rightarrow \frac{4Z^2\alpha^2}{16k^2}\frac{\cos^2\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}\cdot\left(167\cdot10^{-13}MeV\cdot cm\right)^2$$

$$= 2.58\cdot10^{-36}cm^2\frac{\cos^2\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}$$

$|F(q)|^2$ is calculated in the equation from part 2:

$$F(q) = \frac{3}{(qR)^3}\left(-qR\cos(qR)+\sin(qR)\right)$$

$$F(q)^2 = \frac{9}{(qR)^6}\left(-qR\cos(qR)+\sin(qR)\right)^2$$

$$qR = 2Rk\sin\frac{\theta}{2} = 2\left(\frac{3.6\,fm\cdot946MeV}{\hbar c = 197MeV\cdot fm}\right)\sin\frac{\theta}{2} = 34.5746\sin\frac{\theta}{2}$$

Angle	**Calculated** $\left(\frac{d\sigma}{d\Omega}\right)_{Mott}$	**Calculated** $\left|F\left(q^2\right)\right|^2$	**Calculated** $\left(\frac{d\sigma}{d\Omega}\right)$	**Povh 5.7**
20°	$8.3\cdot10^{-35}\ \frac{cm^2}{sr}$	0.00702	$5.8\cdot10^{-37}\ \frac{cm^2}{sr}$	$6\cdot10^{-30}\ \frac{cm^2}{sr}$
25°	$5.2\cdot10^{-35}\ \frac{cm^2}{sr}$	0.00016	$8.6\cdot10^{-39}\ \frac{cm^2}{sr}$	$2\cdot10^{-30}\ \frac{cm^2}{sr}$
30°	$3.6\cdot10^{-35}\ \frac{cm^2}{sr}$	0.0012	$4.5\cdot10^{-38}\ \frac{cm^2}{sr}$	$9\cdot10^{-33}\ \frac{cm^2}{sr}$
35°	$2.6\cdot10^{-35}\ \frac{cm^2}{sr}$	0.00018	$4.7\cdot10^{-39}\ \frac{cm^2}{sr}$	$10^{-32}\ \frac{cm^2}{sr}$
40°	$1.9\cdot10^{-35}\ \frac{cm^2}{sr}$	0.00029	$5.7\cdot10^{-39}\ \frac{cm^2}{sr}$	$5\cdot10^{-34}\ \frac{cm^2}{sr}$
45°	$1.5\cdot10^{-35}\ \frac{cm^2}{sr}$	0.00016	$2.43\cdot10^{-39}\ \frac{cm^2}{sr}$	$3\cdot10^{-35}\ \frac{cm^2}{sr}$
50°	$1.2\cdot10^{-35}\ \frac{cm^2}{sr}$	0.000052	$6.2\cdot10^{-40}\ \frac{cm^2}{sr}$	$9\cdot10^{-36}\ \frac{cm^2}{sr}$
55°	$9.5\cdot10^{-36}\ \frac{cm^2}{sr}$	0.00012	$1.2\cdot10^{-39}\ \frac{cm^2}{sr}$	$7\cdot10^{-37}\ \frac{cm^2}{sr}$
60°	$7.7\cdot10^{-36}\ \frac{cm^2}{sr}$	$4.1\cdot10^{-7}$	$3.2\cdot10^{-42}\ \frac{cm^2}{sr}$	$10^{-38}\ \frac{cm^2}{sr}$

These results are pretty far off, but one will notice that this answer is extraordinarily sensitive to initial conditions. The diffuse surface of Calcium-40 (Povh figure 5.8) is one source of ambiguity; while the behavior is roughly similar to the behavior I might expect, I have to say that to produce a result sensitive to initial conditions with an estimate from a graph is not a good idea.