Ben Sauerwine
Nuclear and Particle Physics Homework 3

1) **Many particle detectors use Silicon micro-strip detectors to measure the trajectories of charged particles. A typical detector is 300 $\mu$m of silicon. Find an expression for a the change in angle of a high-energy $\beta \approx 1$ particle of momentum p GeV due to multiple Coulomb scattering as it passes through the silicon wafer.**

PDG 27.12: (http://pdg.lbl.gov/2006/reviews/passagerpp.pdf):

$$\theta_0 = \frac{13.6mrad}{\beta cp} z\sqrt{\frac{x}{X_0}}\left[1+0.038\ln\left(\frac{x}{X_0}\right)\right]$$

$$X_0(Si) = .0936m$$

$$x = 300\mu m = 0.003m$$

$$c \to 1$$

$$\to \frac{13.6mrad}{1000\,p(GeV)} z\sqrt{\frac{0.0003}{0.0936}}\left[1+0.038\ln\left(\sqrt{\frac{0.0003}{0.0936}}\right)\right]$$

$$\theta_0 = \frac{0.00069z \cdot mrad}{p(GeV)}$$

Z is the charge number of the incident particle.

*Note: Neither my answer 1 or 2 agrees with the professor's answer… however, substituting the numbers he has on his solution doesn't even give his answer!*

2) **Let's take a simple example of a detector to measure an electron scattering angle. We consider a spherical detector at a distance R from the scattering of a 50 GeV electron beam. The detector can measure the position of the electron to an accuracy of $\sigma$. So, the larger we make R, the more accurately we would seem to be able to measure an angle, since the error in $\theta$ is $\frac{\sigma}{R}$.**

   **But, if we imagine the electron traveling through air, then multiple scattering will change the direction also. Calculate the approximate value of R (as a function of $\sigma$) for which the error due to multiple scattering changing the angle and the measurement error from $\sigma$ are comparable. What is the value of R for $\sigma$ = 1 cm?**

$$X_0(air) = 30420cm$$

As in part 1 (note that PDG warns that accuracy of this formula becomes questionable at $\frac{x}{X_0} > 100$):

$$\theta_0 = \frac{13.6mrad}{\beta cp} z\sqrt{\frac{x}{X_0}}\left[1 + 0.038\ln\left(\frac{x}{X_0}\right)\right]$$
$$X_0(air) = 30420cm$$
$$z = 1$$
$$\beta c \to 1$$
$$\to \frac{13.6mrad}{1000\,p(GeV)}\sqrt{\frac{R}{30420}}\left[1 + 0.038\ln\left(\sqrt{\frac{R}{30420}}\right)\right]$$
$$Average \quad Scattering \quad Angle: \Delta y = R\psi = \frac{R\theta_0}{\sqrt{3}} = \sigma$$
$$Solve: \frac{\sqrt{3}\sigma}{R} = \theta_0 = \frac{13.6mrad}{1000\,p(GeV)}\sqrt{\frac{R}{30420}}\left[1 + 0.038\ln\left(\sqrt{\frac{R}{30420}}\right)\right]$$
$$Assume: 0.038\ln\left(\sqrt{\frac{R}{30420}}\right) \approx 0$$
$$\sigma = \frac{13.6mrad}{1000\,p(GeV)\sqrt{3}\sqrt{30420}} R^{\frac{3}{2}}$$
$$R = \left[\frac{1000\,p(GeV)\sqrt{3}\sqrt{30420}}{13.6mrad}\sigma\right]^{\frac{2}{3}}$$
$$\sigma = 1cm : R = \left[\frac{1000\,p(GeV)\sqrt{3}\sqrt{30420}}{13.6mrad}\right]^{\frac{2}{3}} = 790.\frac{cm}{GeV^{\frac{2}{3}}}\left[p(GeV)\right]^{\frac{2}{3}}$$

*Note: Neither my answer 1 or 2 agrees with the professor's answer… however, substituting the numbers he has on his solution doesn't even give his answer!*

3) **Consider a photon pair-converting in the presence of a lead nucleus:** $\gamma Pb \to e^+e^- Pb$**.**

a) **What is the approximate threshold (i.e., minimum possible $E_\gamma$) for this reaction to occur?**

A reasonable guess would be the sum of the rest masses of the product particles, $2 \cdot 0.511MeV = 1.022MeV$, but this does not take into account the conservation of momentum.

**b) Assuming for now that no energy is transferred to the nucleus, determine how much 3-momentum is transferred to the nucleus for a photon at pair conversion threshold.**

$$m_{Pb} \approx 931 \cdot 206 MeV = 191786 MeV$$

From momentum conservation:

$$\begin{bmatrix} E_{\min} \\ E_{\min} \end{bmatrix} + \begin{bmatrix} m_{Pb} \\ 0 \end{bmatrix} = 2\begin{bmatrix} \sqrt{m_e^2 + p_e^2} \\ p_e \end{bmatrix} + \begin{bmatrix} E_{Pb} \\ p_{pb} \end{bmatrix}$$

$$E_{\min} \geq \sqrt{m_e^2 + p_e^2}$$

$$Thus:$$

$$E_{\min} = \sqrt{m_e^2 + p_e^2} + \Delta E_{Pb} =$$

$$2p_e = E_{\min} - p_{pb}$$

$$E_{\min} = 2\sqrt{m_e^2 + \frac{1}{4}\left(E_{\min} - p_{pb}\right)^2} + \Delta E_{Pb}$$

$$E_{\min}{}^2 = 4m_e^2 + \left(E_{\min} - p_{pb}\right)^2$$

$$0 = 4m_e^2 - 2p_{pb}E_{\min} + p_{pb}{}^2$$

$$p_{pb} = \frac{2E_{\min} \pm \sqrt{4E_{\min}{}^2 - 16m_e^2}}{2} = E_{\min} \pm \sqrt{E_{\min}{}^2 - 4m_e^2}$$

$$E_{\min} \to 2m_e : p_{pb} \to 2m_e$$

**c) Use your result from part b to show that the energy transfer to the nucleus is indeed negligibly small.**

$$E = \frac{p_{Pb}{}^2}{2m_{Pb}} = \frac{(2m_e)^2}{2m_{Pb}} = \frac{2m_e{}^2}{m_{Pb}} = \frac{2(0.511)^2}{191786} MeV = little$$

**d) If $E_\gamma$ is about 10% above the threshold, what will the typical energy of the electron or positron be? What momentum does this correspond to?**

$$(1.10)E = 2\sqrt{m_e^2 + p_e^2} = (1.10)(0.511) = 0.5621 MeV$$

$$p_e = m_e\sqrt{(1.10)^2 - 1}$$

$$p_e = 0.23417 \frac{MeV}{c}$$

4) **Consider a system of particles with four-momenta $\vec{p}_i$ obeying $\vec{p}_i \cdot \vec{p}_i = m_i^2$ (That is, the i-th particle has mass $m_i$.)**

a) **Prove that the magnitude M of the total four-momentum $P \equiv \sum_p \vec{p}_i$ must satisfy $M^2 \geq \left( \sum_i m_i \right)^2$.**

Observe that $\vec{p}_i \cdot \vec{p}_i = m_i^2$ is Lorentz-invariant, indicating that this is always true irrespective of frame.

$$\vec{P}^2 = M^2 = \sum_{i,j} \vec{p}_i \cdot \vec{p}_j = \sum_i \vec{p}_i \cdot \vec{p}_i + \sum_{i \neq j} \vec{p}_i \cdot \vec{p}_j = \sum_i m_i^{\ 2} + \sum_{i \neq j} \vec{p}_i \cdot \vec{p}_j$$

It remains, then, to show that $\sum_{i \neq j} \vec{p}_i \cdot \vec{p}_j$ is always positive, irrespective of choice of four-momenta. Note first that the dot product of momentum four-vectors is Lorentz invariant. Then, in the frame of either $\vec{p}_i$ or $\vec{p}_j$, $\vec{p}_i \cdot \vec{p}_j = E_i E_j$ (The special case of two photons will be considered in a moment). Energy is defined to be non-negative or zero for a photon, since a negative energy would imply a negative mass in the particle's rest frame. Thus, $\vec{p}_i \cdot \vec{p}_j = E_i E_j > 0$ for all four-momenta, where these particular energies are taken to be in the rest frame of either particle and this dot-product is Lorentz invariant.

Taking a moment to consider the case of two photons, I notice that these have no contribution to the mass at all. In the case of photons, since all of the energy is tied up in the particle's momentum portion:

$$\vec{p}_{\gamma:i} \cdot \vec{p}_{\gamma:j} = E_i E_j - E_i E_j \cos\theta = E_i E_j (1 - \cos\theta) \geq 0$$

So I see that this dot product is definitely at least zero for both interactions between photons and matter (first paragraph) and also between photons and photons.

Then:

$$\vec{P}^2 = M^2 = \sum_{i,j} \vec{p}_i \cdot \vec{p}_j = \sum_i m_i^{\ 2} + \sum_{i \neq j} \vec{p}_i \cdot \vec{p}_j$$

$$\sum_{i \neq j} \vec{p}_i \cdot \vec{p}_j = E_i E_j (either \quad i,j \quad rest \quad frame) > 0 \quad (Lorentz \quad Inv.)$$

$$\therefore \vec{P}^2 = \sum_i m_i^{\ 2} + \sum_{i \neq j} \vec{p}_i \cdot \vec{p}_j \geq \sum_i m_i^{\ 2}$$

b) **When does $M = \sum_i m_i$?**

The total mass of the system equals the total mass of the components when each component is at the same velocity in the same frame.

5) **A photon cannot pair convert in isolation. However, the other body that 'assists' in energy-momentum conservation need not be massive like a nucleus. Let's investigate pair production near a free electron:** $\gamma e^- \rightarrow e^+ e^- e^-$ **.**

a) **Assuming the initial electron is at rest, describe qualitatively the motion of the three final-state particles for pair production near the threshold energy.**

The newly-produced electron, the positron and the nucleus will all be at rest with respect to each other, but will be moving with a new momentum in the lab frame.

b) **For the case of the initial electron at rest, calculate the threshold energy (the energy of the photon for this process to be energetically allowed).**

Define: Rest mass of electrons and positron $m_e$, threshold energy $E_{\min}$.

At the threshold energy, taking all momenta to be in the x-direction, (e.g. $\begin{bmatrix} E \\ p_x \end{bmatrix}$) I have

$$\begin{bmatrix} E_{\min} \\ E_{\min} \end{bmatrix} + \begin{bmatrix} m_e \\ 0 \end{bmatrix} = 3\begin{bmatrix} \sqrt{m_e^2 + p_{products}^2} \\ p_{products} \end{bmatrix}$$

$$Thus: E_{\min} = 3p_{products}$$

$$\left(\begin{bmatrix} E_{\min} \\ E_{\min} \end{bmatrix} + \begin{bmatrix} m_e \\ 0 \end{bmatrix}\right)^2 = \left(3\begin{bmatrix} \sqrt{m_e^2 + \frac{1}{3}E_{\min}^2} \\ \frac{1}{3}E_{\min} \end{bmatrix}\right)^2$$

$$2E_{\min}m_e + m_e^2 = 9\left[m_e^2 + \frac{1}{9}E_{\min}^2 - \frac{1}{9}E_{\min}^2\right] = 9m_e^2$$

$$2E_{\min}m_e = 8m_e^2$$

$$E_{\min} = 4m_e$$