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Nuclear and Particle Physics Homework 2

1) What is the lowest-Z element for which the energy of the K_α X-ray exceeds 1 keV?

The K_α transition corresponds to the transition of a hole from the innermost orbital, 1S, to the next further level, 2S. This relation comes from Mosely's Law, a brief explanation of which is given:

$$E_{K_\alpha} = \left(\frac{3}{4}\right)(13.6eV)(Z-1)^2$$

The first portions come from the basic energy difference between the 1S and 2S levels of a Hydrogen atom: $\left[1-\frac{1}{2^2}\right](13.6eV)$. The $(Z-1)^2$ comes from the additional Coulomb contribution of the nucleus, minus one electron shielding from the K shell (the other one is occupied by the hole).

$$\sqrt{\left(\frac{4}{3}\right)\frac{1000}{13.6}}+1 = Z$$
$$Z = 10.90 \rightarrow 11 \rightarrow Sodium$$

2) The discovery of the neutron was announced in 1932 by Chadwick. The original paper quoted the approximate maximum (i.e., head-on collision) recoil velocities for hydrogen and nitrogen atoms struck by neutrons:

$$v_H = 3.3\cdot 10^9 \frac{cm}{s} \quad v_N = 4.7\cdot 10^8 \frac{cm}{s}$$

Use this data to calculate the mass of the neutron in units of the proton (hydrogen) mass.

I use a non-relativistic approximation, since these velocities are on the order of 0.01c. Further, assume that the neutrons come from a constant-velocity emitter (but with this velocity unknown) and assume that scattering is totally elastic in one dimension:

$$m_H v_H + m_n v_{F,H,n} = m_n v_{I,n}$$
$$m_N v_N + m_n v_{F,N,n} = m_n v_{I,n}$$
$$\frac{1}{2}m_H {v_H}^2 + \frac{1}{2}m_n {v_{F,H,n}}^2 = \frac{1}{2}m_n {v_{I,n}}^2$$
$$\frac{1}{2}m_N {v_N}^2 + \frac{1}{2}m_n {v_{F,N,n}}^2 = \frac{1}{2}m_n {v_{I,n}}^2$$

$define \quad r = \frac{m_n}{m_H}, \quad R = \frac{m_n}{m_N}$. Now:

$$\frac{1}{r} v_H = v_{I,n} - v_{F,H,n}$$

$$\frac{1}{r} v_H{}^2 = v_{I,n}{}^2 - v_{F,H,n}{}^2$$

$Divide:$

$$v_H = v_{I,n} + v_{F,H,n}$$

$$v_H = v_{I,n} + \left(v_{I,n} - \frac{1}{r} v_H \right)$$

$$v_H = \frac{2}{1 + \frac{1}{r}} v_{I,n}$$

Similarly:

$$v_N = \frac{2}{1 + \frac{1}{R}} v_{I,n}$$

Note: This calculation does NOT work properly if you assume, a priori, that

$$R = \frac{m_n}{m_N} = \frac{m_n}{7m_H + 7m_n} = \frac{1}{7} \frac{m_n}{m_H + m_n} = \frac{1}{7} \left(\frac{1}{r} + 1 \right)^{-1}$$ **[DOES NOT WORK!]**

This problem took me forever because of this single, seemingly plausible formula**!**

However, if you let:

$$\frac{r}{R} = \frac{\frac{m_n}{m_H}}{\frac{m_n}{m_N}} = 14:$$

And so:

$$\frac{v_N}{v_H} = \frac{1 + \frac{1}{r}}{1 + \frac{1}{R}} = \frac{r + 1}{1 + \frac{r}{R}} = \frac{1 + r}{15}$$

$$r = 15 \frac{v_N}{v_H} - 1 = 1.14 = \frac{m_n}{m_H}$$

3) **Later in the paper, a more accurate mass determination was presented. This led Chadwick to believe that the neutron had about a 1 MeV binding energy due to an apparent mass deficit. That is, he thought**

$m_n - (m_p + m_e) \approx -1MeV$ **, as opposed to the current best value of** $+0.7782MeV$**.**

Treated as point charges, how far apart would a bound proton and electron have to be to account for a 1 MeV binding energy?

$$\frac{1}{2}\left[\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}\right] = 1000000 \cdot \left(e\frac{J}{eV}\right)$$

$$r = \frac{1}{2 \cdot 1000000}\left[\frac{e}{4\pi\varepsilon_0}\right] = 7.19003 \cdot 10^{-16}\,m = 0.719\,fm$$

4) **(Perkins 1.4) Deduce an expression for the energy of a** γ**-ray from the decay of the neutral pion,** $\pi^0 \to 2\gamma$**, in terms of the mass m, the energy E, and the velocity** βc **of the pion and the angle of emission** θ **in the pion rest frame. Show that if the pion has spin zero, so that the angular distribution is isotropic, the laboratory energy spectrum of the** γ**-rays will be flat, extending from** $E\frac{1+\beta}{2}$ **to** $E\frac{1-\beta}{2}$**. Find an expression for the disparity D (the ratio of energies) of the** γ**-rays and show that D > 3 in half the decays and D > 7 in one quarter of them.**

First, consider the decay in the rest frame of the pion. I will Lorentz boost the photons later. Let $E_{rest} = mc^2$, so that he energy of each photon will be $\frac{E_{rest}}{2}$ with a momentum $P = \frac{E_{rest}}{c}$ in their respective, opposite directions. Now I boost into the Lab frame:

$$\vec{P}_1 = \frac{E_{rest}}{2}\begin{bmatrix}1\\ \cos\theta\\ \sin\theta\\ 0\end{bmatrix} \quad \vec{P}_2 = \frac{E_{rest}}{2}\begin{bmatrix}1\\ -\cos\theta\\ -\sin\theta\\ 0\end{bmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\Lambda\vec{P}_1 = \frac{E_{rest}}{2}\begin{bmatrix}\gamma & \gamma\beta & 0 & 0\\ \gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ \cos\theta\\ \sin\theta\\ 0\end{bmatrix} = \frac{\gamma E_{rest}}{2}\begin{bmatrix}1+\beta\cos\theta = E_1\\ etc.\\ etc.\\ etc.\end{bmatrix} \quad \Lambda\vec{P}_2 = \frac{\gamma E_{rest}}{2}\begin{bmatrix}1-\beta\cos\theta = E_2\\ etc.\\ etc.\\ etc.\end{bmatrix}$$

Above, since I have not yet defined E, define $E = E_{rest}\gamma = mc^2\gamma$, as expected.

Now assuming angular isotropy, e.g., $\frac{\partial N_{released}}{\partial \cos\theta} = \frac{1}{2}$ so that $\int_{-1}^{1} \frac{\partial N_{released}}{\partial \cos\theta} d\cos\theta = 1$. Now, $\frac{\partial N_{released}}{\partial E} = \frac{\partial N_{released}}{\partial \cos\theta}\frac{\partial \cos\theta}{\partial E} = \frac{1}{2}\left[\frac{2}{E\beta}\right] = \frac{1}{E\beta}$, and so it is clear that the distribution is uniform with $\int_{\frac{E}{2}(1-\beta)}^{\frac{E}{2}(1+\beta)} \frac{1}{E\beta} dx = 1$ for normality.

Finally, in terms of the ratio of the disparities between the two energies, I have:

$$D = \frac{1+\beta\cos\theta}{1-\beta\cos\theta}.$$

For the final verifications, let $\beta \approx 1$. Then,

$$D_{\beta\to 1} = \frac{1+\cos\theta}{1-\cos\theta}.$$

$D_{\beta\to 1} > 3$ when $\cos\theta > \frac{1}{2} \quad or \quad \theta < \frac{\pi}{3}$. Since the angular distribution of particles is isotropic, then, I may take:

$$\frac{2}{4\pi}\int_0^{\frac{\pi}{3}}\int_0^{2\pi} \sin\theta d\theta d\phi = \int_0^{\frac{\pi}{3}} \sin\theta d\theta = -\cos\frac{\pi}{3} + \cos 0 = \frac{1}{2},$$

e.g. half of the particles land in the region where D > 3.

For $D_{\beta\to 1} > 7$, I have $\cos\theta > \frac{3}{4} \quad or \quad \theta < ArcCos\frac{3}{4}$. Again, the angular distribution is isotropic for:

$$\frac{2}{4\pi}\int_0^{ArcCos\left[\frac{3}{4}\right]}\int_0^{2\pi} \sin\theta d\theta d\phi = \int_0^{ArcCos\left[\frac{3}{4}\right]} \sin\theta d\theta = -\cos\left(ArcCos\left(\frac{3}{4}\right)\right) + \cos 0 = -\frac{3}{4} + 1 = \frac{1}{4},$$

e.g., a quarter of the particles land in the D > 7 region.

5) **Consider the decay of the Lambda particle to a proton and a charged pion $\Lambda \to p\pi^-$ in the Lambda center-of-mass frame.**

a) What is the Q-value (energy release) of this decay?

Λ rest mass: $1115.6\frac{MeV}{c^2}$

p rest mass: $938.279\frac{MeV}{c^2}$

π^- rest mass: $139.6\frac{MeV}{c^2}$

$Q = 1115.6 - 938.3 - 139.6 = 37.7 MeV$

b) What is the magnitude of the momentum for each of the two decay products?

In terms of invariant mass, I have:

$\Lambda \to p\pi^-$

Now in terms of four-vectors:

$$\Lambda^2 = \left(p + \pi^-\right)^2$$
$$m_\Lambda^2 = m_p^2 + m_\pi^2 + 2p\pi^-$$
$$\frac{m_\Lambda^2 - m_p^2 - m_\pi^2}{2} = p\pi^- = E_p E_\pi - p_p p_\pi \cos 180^\circ$$

The 180-degrees must arise from conservation of momentum: in fact, the momenta must be equal and opposite!

$$\frac{m_\Lambda^2 - m_p^2 - m_\pi^2}{2} = E_p E_\pi + p^2 = E_p E_\pi + E_p^2 - m_p^2$$
$$\frac{m_\Lambda^2 + m_p^2 - m_\pi^2}{2} = E_p\left(E_\pi + E_p\right) = E_p m_\Lambda$$
$$\frac{m_\Lambda^2 + m_p^2 - m_\pi^2}{2m_\Lambda} = E_p$$
$$\frac{1115.6^2 + 938.3^2 - 139.6^2}{2 \cdot 1115.6} = E_p = 943.6 MeV$$
$$943.6^2 - 938.3^2 = \left(100.384 MeV\right)^2$$
$$100.384 MeV = p_p$$