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Nuclear and Particle Physics Homework 10

1) **Consider the decay of a particle of mass M to particles of mass m_1, m_2, m_3.**

a) **Use 4-momentum conservation to show that** $m_{12}^2 + m_{13}^2 + m_{23}^2 = M^2 + m_1^2 + m_2^2 + m_3^2$**.**

$$m_{ij} = (p_i + p_j)^2$$

Expanding this with the definition, I have:

$$m_{12}^2 + m_{13}^2 + m_{23}^2 = 2p_1^2 + 2p_2^2 + 2p_3^2 + 2p_1 \cdot p_2 + 2p_1 \cdot p_3 + 2p_2 \cdot p_3$$

Note that by 4-momentum conservation, I have:

$$(p_1 + p_2 + p_3)^2 = M^2 = p_1^2 + p_2^2 + p_3^2 + 2p_1 \cdot p_2 + 2p_1 \cdot p_3 + 2p_2 \cdot p_3$$

Using this identity, then, I have:

$$m_{12}^2 + m_{13}^2 + m_{23}^2 = (p_1 + p_2 + p_3)^2 + p_1^2 + p_2^2 + p_3^2 = M^2 + m_1^2 + m_2^2 + m_3^2$$

Just as expected.

b) **Show that** $E_3 = \frac{M^2 + m_3^2 - m_{12}^2}{2M}$**, where** $m_{ij} = (p_i + p_j)^2$ **and** E_3 **is the energy of particle 3 in the parent rest frame.**

$$P = \begin{pmatrix} M \\ \vec{0} \end{pmatrix} \quad p_i = \begin{pmatrix} E_i \\ \vec{p}_i \end{pmatrix}$$

$$E_3 = \frac{P \cdot p_3}{M} = \frac{(p_1 + p_2 + p_3) \cdot p_3}{M} = \frac{p_1 \cdot p_3 + p_2 \cdot p_3 + m_3^2}{M} = \frac{2p_1 \cdot p_3 + 2p_2 \cdot p_3 + 2m_3^2}{2M}$$

$$= \frac{m_{13}^2 - m_1^2 - m_3^2 + m_{23}^2 - m_2^2 - m_3^2 + 2m_3^2}{2M} = \frac{m_{13}^2 + m_{23}^2 - m_1^2 - m_2^2}{2M} = \frac{M^2 - m_{12}^2 + m_3^2}{2M}$$

c) **Evaluate** $(P \cdot p_3)(p_1 \cdot p_2)$ **in terms of** M, m_i, m_{ij} **only.**

$$(P \cdot p_3) = \frac{M^2 - m_{12}^2 + m_3^2}{2} \quad (p_1 \cdot p_2) = \frac{m_{12}^2 - m_1^2 - m_2^2}{2}$$

$$(P \cdot p_3)(p_1 \cdot p_2) = \left(\frac{M^2 - m_{12}^2 + m_3^2}{2}\right)\left(\frac{m_{12}^2 - m_1^2 - m_2^2}{2}\right)$$

d) **Show that** $4M^2 dE_1 dE_2 = dm_{12}^2 dm_{23}^2$

$$E_1 = \frac{M^2 - m_{23}^2 + m_1^2}{2M} \quad E_2 = \frac{M^2 - m_{13}^2 + m_2^2}{2M}$$

$$dE_1 = \frac{-dm_{23}^2}{2M} \quad dE_2 = \frac{-dm_{13}^2}{2M}$$

$$4M^2 dE_1 dE_2 = 4M^2 \left(\frac{-dm_{23}^2}{2M}\right)\left(\frac{-dm_{13}^2}{2M}\right) = dm_{23}^2 dm_{13}^2$$

Since $m_{12}^2 = C - m_{23}^2 - m_{13}^2$, the Jacobian is one and therefore any pair of $dm_{12}^2, dm_{23}^2, dm_{13}^2$ can be used.

2) **The amplitudes for $K \to \pi\pi$ decay are:**

$$A(K_S \to \pi^+\pi^-) = \sqrt{\frac{2}{3}} A_0 + \sqrt{\frac{1}{3}} A_2$$

$$A(K_S \to \pi^0\pi^0) = \sqrt{\frac{1}{3}} A_0 - \sqrt{\frac{2}{3}} A_2$$

$$A(K^+ \to \pi^+\pi^0) = \sqrt{\frac{3}{2}} A_2$$

expressed in terms of amplitudes A_0 and A_2 for $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$ transitions, respectively.

Use experimental data from the PDG to estimate a value for $\left|\frac{A_2}{A_0}\right|$. Useful data include the K^+ and K_S lifetimes, and the branching ratios $B_i = \frac{\Gamma_i}{\Gamma}$. Please state any assumptions you make when obtaining this estimate, and argue that they are accurate enough (within a few percent). Finally, note that the amplitudes may have a relative phase between them! Do not assume it is zero.

From the PDG:
K^+: http://pdg.lbl.gov/2006/listings/s010.pdf
K_S: http://pdg.lbl.gov/2006/listings/s012.pdf

I find:

$$K_S \quad Lifetime: 0.8953 \cdot 10^{-10}\, s = \frac{1}{\Gamma_{S,tot}}$$

$$K^+ \quad Lifetime: 1.2385 \cdot 10^{-8}\, s = \frac{1}{\Gamma_{+,tot}}$$

$$B\left(K_S \to \pi^0\pi^0\right) = \frac{\Gamma\left(K_S \to \pi^0\pi^0\right)}{\Gamma} = 30.69\% = \frac{\Gamma_{S,\pi^0\pi^0}}{\Gamma_{S,tot}}$$

$$B\left(K_S \to \pi^+\pi^-\right) = \frac{\Gamma\left(K_S \to \pi^+\pi^-\right)}{\Gamma} = 69.20\% = \frac{\Gamma_{S,\pi^+\pi^-}}{\Gamma_{S,tot}}$$

$$B\left(K^+ \to \pi^+\pi^0\right) = \frac{\Gamma\left(K^+ \to \pi^+\pi^0\right)}{\Gamma} = 20.92\% = \frac{\Gamma_{+,\pi^+\pi^0}}{\Gamma_{+,tot}}$$

Solving,

$$\Gamma_{S,\pi^0\pi^0} = 3.430 \cdot 10^{11}$$

$$\Gamma_{S,\pi^+\pi^-} = 7.729 \cdot 10^{11}$$

$$\Gamma_{+,\pi^+\pi^0} = 1.689 \cdot 10^{11}$$

I have a relationship: $\mathrm{M} = \left\langle f \left| H_{\text{int}} \right| i \right\rangle \quad \Gamma = \frac{1}{8\pi} \left|\mathrm{M}\right|^2 \frac{\left|p^*\right|}{M^2}$

However, these constant factors are the same for each situation here, so that effectively:

$$(1): \frac{2}{3}\left|A_0\right|^2 + \frac{1}{3}\left|A_2\right|^2 + \sqrt{\frac{2}{9}} A_2 A_0^+ + \sqrt{\frac{2}{9}} A_2^+ A_0 = C\frac{\Gamma_{S,\pi^+\pi^-}}{\Gamma_{S,tot}}$$

$$(2): \frac{1}{3}\left|A_0\right|^2 + \frac{2}{3}\left|A_2\right|^2 - \sqrt{\frac{2}{9}} A_2 A_0^+ - \sqrt{\frac{2}{9}} A_2^+ A_0 = C\frac{\Gamma_{S,\pi^0\pi^9}}{\Gamma_{S,tot}}$$

$$(3): \frac{3}{2}\left|A_2\right|^2 = C\frac{\Gamma_{+,\pi^+\pi^0}}{\Gamma_{+,tot}}$$

$$add \quad (1)+(2):$$

$$\left|A_0\right|^2 + \left|A_2\right|^2 = C\frac{\Gamma_{S,\pi^+\pi^-}}{\Gamma_{S,tot}} + C\frac{\Gamma_{S,\pi^0\pi^9}}{\Gamma_{S,tot}}$$

Solving,

$$\left|A_0\right|^2 = C\frac{\Gamma_{S,\pi^+\pi^-}}{\Gamma_{S,tot}} + C\frac{\Gamma_{S,\pi^0\pi^0}}{\Gamma_{S,tot}} - \frac{2}{3}C\frac{\Gamma_{+,\pi^+\pi^0}}{\Gamma_{+,tot}}$$

$$\left|A_2\right|^2 = \frac{2}{3}C\frac{\Gamma_{+,\pi^+\pi^0}}{\Gamma_{+,tot}}$$

$subtract \quad (1)-(2):$

$$\sqrt{\frac{8}{9}}A_2 A_0^+ + \sqrt{\frac{8}{9}}A_2^+ A_0 = C\frac{\Gamma_{S,\pi^+\pi^-}}{\Gamma_{S,tot}} - C\frac{\Gamma_{S,\pi^0\pi^9}}{\Gamma_{S,tot}}$$

$$\frac{|A_0|}{|A_2|}\left(e^{i\phi_2}e^{-i\phi_0} + e^{-i\phi_2}e^{i\phi_0}\right) = \sqrt{\frac{9}{8}}\frac{C\Gamma_{S,\pi^+\pi^-} - C\Gamma_{S,\pi^0\pi^9}}{|A_2|^2\Gamma_{S,tot}}$$

$$2\left|\frac{A_0}{A_2}\right| = \sqrt{\frac{9}{8}}\frac{C\Gamma_{S,\pi^+\pi^-} - C\Gamma_{S,\pi^0\pi^9}}{|A_2|^2\Gamma_{S,tot}}$$

$$\left|\frac{A_0}{A_2}\right| = \frac{3}{4}\sqrt{\frac{9}{8}}\frac{\Gamma_{S,\pi^+\pi^-} - \Gamma_{S,\pi^0\pi^9}}{\Gamma_{+,\pi^+\pi^0}}\frac{\Gamma_{+,tot}}{\Gamma_{S,tot}} = \frac{9}{8}\sqrt{\frac{1}{2}}\frac{\Gamma_{S,\pi^+\pi^-} - \Gamma_{S,\pi^0\pi^9}}{\Gamma_{+,\pi^+\pi^0}}\frac{\Gamma_{+,tot}}{\Gamma_{S,tot}}$$

Specifically, then,

$$\left|\frac{A_0}{A_2}\right| = \frac{3}{4}\sqrt{\frac{9}{8}}\frac{\Gamma_{S,\pi^+\pi^-} - \Gamma_{S,\pi^0\pi^9}}{\Gamma_{+,\pi^+\pi^0}}\frac{\Gamma_{+,tot}}{\Gamma_{S,tot}} = \frac{9}{8}\sqrt{\frac{1}{2}}\frac{\Gamma_{S,\pi^+\pi^-} - \Gamma_{S,\pi^0\pi^9}}{\Gamma_{+,\pi^+\pi^0}}\frac{\Gamma_{+,tot}}{\Gamma_{S,tot}} = 202.8$$

The only assumptions I had to make here originated from the PDG's assumptions in calculating these lifetimes and branching ratios, which are outlined in their white papers. This justifies ignoring the amplitude A_2 compared to that of A_0.

3) **Consider decays of a spinless particle to N spinless decay products. In particular, let's focus on the number of variables needed to describe the final state.**

a) **In a 3-body decay, there are naively sixteen quantities which comprise the four (1 parent, 3 daughters) 4-vectors. Explain why fourteen of these degrees of freedom are either constrained or not relevant to the structure of the final state, leaving the two familiar Dalitz plot variables.**

(i) 4 constraints come from energy and momentum conservation.
(ii) 4 constraints come from the constraint $E^2 = p^2 + m^2$
(iii) From frame invariance, I may select my initial condition as $E = M \quad \vec{P} = \vec{0}$. This gives only 3 constraints, since one is redundant from (ii)
(iv) Orientation gives 3 constraints, as I may fix one decay to be in the z-direction and another to be in the xz-plane.

Thus, I get 14 constraints.

b) **Generalize the counting in (a) to determine how many Dalitz variables there would be for an N-body final state.**

Since none of these eliminations depends on the number of particles and the mass for the particles is known, the number should be $3N-7$: an additional 3 constraints for the momentum of each particle.

c) **Finally, for a 2-body decay, there are zero variables describing the structure of the final state, but the formula for general N would predict -1. What changed that explains this apparent discrepancy?**

Momentum conservation ensures that the orientation of the decay will be entirely along the same line, and so one of orientation's constraints becomes redundant.

4) **One general expression for n-body decay which uses a general expression for phase space is $d\Gamma = \frac{(2\pi)^4}{2M}|\mathrm{M}|^2 d\Phi_n(P; p_1,..., p_n)$, where the phase-space term is $\Phi_n(P; p_1,..., p_n) = \delta^4\left(P - \sum_i p_i\right)\prod_i \frac{d^3 p_i}{(2\pi)^3 2E_i}$. Here, P is the four-momentum of the parent of mass M which decays into n particles of four-momentum p_i (but in $d^3 p_i$ we refer only to the three-momenta).**

a) **Show that this gives the formula $d\Gamma = \frac{1}{32\pi^2}\frac{|p^*|}{M^2}|\mathrm{M}|^2 d\Omega$ for 2-particle decay.**

Thanks to Prof. Roy Briere for providing elegant solutions, whose procedure I follow here.

p^* is the center of mass momentum.

An initial substitution gives

$$d\Gamma = \frac{1}{8(2\pi)^2 M}|\mathrm{M}|^2 \delta^4(P - p_1 - p_2)\frac{d^3 p_1 d^3 p_2}{E_1 E_2}$$

$$Take \quad CM \quad Frame : (M, \vec{0})$$

$$= \frac{1}{8(2\pi)^2 M}|\mathrm{M}|^2 \delta(M - E_1 - E_2)\delta^3(-\vec{p}_1 - \vec{p}_2)\frac{d^3 p_1 d^3 p_2}{E_1 E_2}$$

Now the delta function ensures that $\vec{p}_1 = -\vec{p}_2$: Then, $\int d^3 p_2 \delta^3(-\vec{p}_1 - \vec{p}_2) = 1$ and:

$$d\Gamma = \frac{1}{8(2\pi)^2 M}|\mathrm{M}|^2 \delta(M - E_1 - E_2)\frac{d^3 p_1}{E_1 E_2}$$

$$d^3 p_1 = p_1^2 dp_1 d\Omega$$

$$d\Gamma = \frac{|\mathrm{M}|^2 d\Omega}{8(2\pi)^2 M}\int \frac{p_1^2 dp_1}{E_1 E_2}\delta(M - E_1 - E_2)$$

The integral over this delta function is then given by:

$$\int \frac{p_1^2 dp_1}{E_1 E_2}\delta(M - E_1 - E_2) = \left[\frac{p_1^2 dp_1}{E_1 E_2}\frac{1}{\left|\left.\frac{\partial(M - E_1 - E_2)}{\partial p_1}\right|_{M - E_1 - E_2 = 0}\right|}\right]_{M - E_1 - E_2 = 0}$$

Given the above argument, then:

$$M - E_1 - E_2 = M - \sqrt{m_1^2 + p_1^2} - \sqrt{m_1^2 + p_1^2}$$

$$\left.\left[\frac{\partial(M - E_1 - E_2)}{\partial p_1}\right]\right|_{M = E_1 + E_2} = \frac{p_1}{E_1} + \frac{p_1}{E_2} = \frac{M p_1}{E_1 E_2}$$

And so on this substitution, I am left with:

$$d\Gamma = \frac{1}{32\pi^2}\frac{p_1}{M^2}|\mathrm{M}|^2 d\Omega$$

Just as expected.

b) Show that this gives $d\Gamma = \frac{1}{(2\pi)^3}\frac{1}{8M}|\mathrm{M}|^2 dE_1 dE_2$ for 3-particle decay.

Substitution gives

$$d\Gamma = \frac{1}{16M(2\pi)^5}|\mathrm{M}|^2 \delta^4(P - p_1 - p_2 - p_3)\frac{d^3 p_1 d^3 p_2 d^3 p_3}{E_1 E_2 E_3}$$

$$= \frac{1}{16M(2\pi)^5}|\mathrm{M}|^2 \delta(M - E_1 - E_2 - E_3)\delta^3(-\vec{p}_1 - \vec{p}_2 - \vec{p}_3)\frac{d^3 p_1 d^3 p_2 d^3 p_3}{E_1 E_2 E_3}$$

Integrating over $\vec{p}_3$ gives:

$$d\Gamma = \frac{1}{16M(2\pi)^5}|\mathrm{M}|^2\delta(M - E_1 - E_2 - E_3)\frac{d^3p_1d^3p_2}{E_1E_2E_3}$$

Next, take

$E^2 = m^2 + p^2 \quad 2EdE = 2pdp$

so that:

$d^3p_1d^3p_2 = p_1p_2E_1E_2d\Omega_1d\Omega_2dE_1dE_2$

Now I have:

$$d\Gamma = \frac{1}{16M(2\pi)^5}|\mathrm{M}|^2\delta(M - E_1 - E_2 - E_3)\frac{p_1p_2}{E_3}d\Omega_1d\Omega_2dE_1dE_2$$

Again, I must evaluate the integrals over $d\Omega_1d\Omega_2 = d(\cos\theta_1)d(\cos\theta_2)d\phi_1d\phi_2$. Taking spatial isotropy, I may consider θ_2 as an offset from θ_1 so that:

$$\begin{aligned}
&\delta(M - E_1 - E_2 - E_3) \\
&= \delta\left(M - E_1 - E_2 - \sqrt{\left((-\vec{p}_1 - \vec{p}_2)^2 + m_3^2\right)}\right) \\
&= \delta\left(M - E_1 - E_2 - \sqrt{p_1^2 + p_2^2 + 2p_1p_2\cos\theta + m_3^2}\right)
\end{aligned}$$

Then,

$$\left|\frac{\partial(M - E_1 - E_2 - E_3)}{\partial\cos\theta}\right|_{M = E_1 + E_2 + E_3} = \frac{p_1p_2}{E_3}$$

so that

$$\int\delta(M - E_1 - E_2 - E_3)\frac{p_1p_2}{E_3}d\cos\theta_2 = 1$$

Finally, $\int d(\cos\theta_1)d\phi_1d\phi_2 = 8\pi^2$ since the remaining parts of the integral are transparent to these due to spatial coordinate system being preferred: Then, .

$$d\Gamma = \frac{1}{8M(2\pi)^3}|\mathrm{M}|^2dE_1dE_2$$