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Numerical Analysis Homework 6

Kincaid 9.2.2) Prove that the stability condition for the method defined by the Crank-Nicolson Method,

$$\frac{\theta}{h^2}\left(v_{i+1,j}-2v_{i,j}+v_{i-1,j}\right)+\frac{1-\theta}{h^2}\left(v_{i+1,j-1}-2v_{i,j-1}+v_{i-1,j-1}\right)=\frac{1}{k}\left(v_{i,j}-v_{i,j-1}\right)$$

is the inequality $s \le (2-4\theta)^{-1}$ if $0 \le \theta < \frac{1}{2}$, but if $\frac{1}{2} \le \theta \le 1$ there is no restriction on s.

Recall that $s=\frac{k}{h^2}$.

Rewriting the above equation a bit, I have:

$$\theta s\left(v_{i+1,j}-2v_{i,j}+v_{i-1,j}\right)+(1-\theta)s\left(v_{i+1,j-1}-2v_{i,j-1}+v_{i-1,j-1}\right)=v_{i,j}-v_{i,j-1}$$

$$\theta s\left(v_{i+1,j}-2v_{i,j}+v_{i-1,j}\right)-v_{i,j}=-(1-\theta)s\left(v_{i+1,j-1}-2v_{i,j-1}+v_{i-1,j-1}\right)-v_{i,j-1}$$

$$\theta s v_{i+1,j}-(2\theta s+1)v_{i,j}+\theta s v_{i-1,j}=-(1-\theta)s v_{i+1,j-1}+(2(1-\theta)s-1)v_{i,j-1}-(1-\theta)s v_{i-1,j-1}$$

Expressing this in matrix form, I have:

$$-\theta s v_{i+1,j}+(2\theta s+1)v_{i,j}-\theta s v_{i-1,j}=(1-\theta)s v_{i+1,j-1}-(2(1-\theta)s-1)v_{i,j-1}+(1-\theta)s v_{i-1,j-1}$$

$$(I+\theta sB)V_j=(I-(1-\theta)sB)V_{j-1}$$

Where $V_j=\begin{bmatrix}\ldots\\ v_{i-1,j}\\ v_{i,j}\\ \ldots\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1 & 0 & 0\ldots\\ -1 & 2 & -1 & 0\ldots\\ 0 & -1 & 2 & -1\ldots\\ 0\ldots & 0\ldots & -1\ldots & 2\ldots\end{bmatrix}$.

Since the eigenvalues of B are $2(1-\cos\phi_i)$ with $(\cos\phi_i \ne \pm 1)$, I see that the condition for stability becomes:

$$\left\|(I+\theta sB)^{-1}(I-(1-\theta)sB)\right\|<1$$

$$\left|(1+2\theta s(1-\cos\phi))^{-1}(1-2(1-\theta)s(1-\cos\phi))\right|<1$$

$$\left|\frac{1-2(1-\theta)s(1-\cos\phi)}{1+2\theta s(1-\cos\phi)}\right|<1$$

Now I notice that for all choices of θ in the range $\frac{1}{2} \le \theta \le 1$, the portion in the absolute value is necessarily positive, and further that the numerator is strictly less than or equal to the denominator:

$$1-2(1-\theta)s(1-\cos\phi) \le 1+2\theta s(1-\cos\phi)$$
$$-2(1-\theta)s(1-\cos\phi) \le 2\theta s(1-\cos\phi)$$
$$-(1-\theta) \le \theta$$
$$\theta - 1 \le \theta$$

This is true for all θ and makes no restriction on s.

However, if $\theta < \frac{1}{2}$, a different situation arises whereby this ratio can drop below zero, and the absolute value can break the inequality. Differentiation shows that the local maxima are at $\phi = n\pi$, where n is an integer. The condition that $(\cos\phi_i \ne \pm 1)$ indicates that in this case I may take the inequality to be $\le$ because the border cases will never be included. Since $\cos\phi_i = 1$ clearly satisfies the inequality, then, I must constrain:

$$\left|\frac{1-4(1-\theta)s}{1+4\theta s}\right| \le 1$$
$$\frac{4(1-\theta)s-1}{1+4\theta s} \le 1$$
$$4(1-\theta)s - 1 \le 1 + 4\theta s$$
$$4s - 8\theta s \le 2$$
$$s \le (2-4\theta)^{-1}$$

just as expected.

Kincaid 9.7.4) Carry out a stability analysis on this numerical procedure:

$$\frac{1}{2k}\left(v_{j,n+1} - v_{j,n-1}\right) = \frac{\alpha}{2h}\left(v_{j+1,n} - v_{j-1,n}\right)$$

For the problem:

$$u_t = \alpha u_x$$
$$u(x,0) = f(x)$$

First, define $s \equiv \frac{k\alpha}{2h}$.

Then, I have:

$$v_{j,n+1} = 2s\left(v_{j+1,n} - v_{j-1,n}\right) + v_{j,n-1}$$

Using a Fourier argument, I take $v_{j,n} = e^{ij\theta h} e^{n\lambda k}$, with $i = \sqrt{-1}$.

Now I have under substitution:

$$e^{ij\theta h} e^{(n+1)\lambda k} = 2s\left(e^{i(j+1)\theta h} - e^{i(j-1)\theta h}\right) e^{n\lambda k} + e^{ij\theta h} e^{(n-1)\lambda k}$$

$$e^{\lambda k} = 2s\left(e^{i\theta h} - e^{-i\theta h}\right) + e^{-\lambda k}$$

$$e^{\lambda k} - e^{-\lambda k} = 4is \sin \theta h$$

Multiplying both sides by their complex conjugate, I have

$$e^{2\lambda k} + e^{-2\lambda k} - 2 = 16s^2 \sin^2 \theta h$$

For stability, I require $\left|e^{\lambda k}\right| \le 1$. Then, I have:

$$\left(e^{\lambda k}\right)^2 = 16s^2 \sin^2 \theta h + 2 - e^{-2\lambda k}$$

$$x \equiv \left|e^{\lambda k}\right|$$

$$x^2 = 16s^2 \sin^2 \theta h + 2 - \frac{1}{x^2}$$

$$x^4 - \left(16s^2 \sin^2 \theta h + 2\right) x^2 + 1 = 0$$

Using Mathematica to find the roots of this equation, one sees that there are two roots that must be considered, each of which must have $x \equiv \left|e^{\lambda k}\right| \le 1$ individually for all θ.

$$x^2 = 1 + 8s^2 \sin^2 \theta h - 4\sqrt{s^2 \sin^2 \theta h + 4s^4 \sin^4 \theta h}$$

$$x^2 = 1 + 8s^2 \sin^2 \theta h + \frac{1}{2}\sqrt{\left(2 + 16s^2 \sin^2 \theta h\right)^2 - 4}$$

Looking at the first one,

$$x^2 = 1 + 8s^2 \sin^2 \theta h - 4\sqrt{s^2 \sin^2 \theta h + 4s^4 \sin^4 \theta h} \le 1 + 8s^2 \sin^2 \theta h - 4\sqrt{4s^4 \sin^4 \theta h} = 1$$

I see then that the first root certainly has $x^2 \le 1$ and so that $x = \left|e^{\lambda k}\right| \le 1$ for all s.

Looking at the second one,

$$x^2 = 1 + 8s^2 \sin^2 \theta h + \frac{1}{2}\sqrt{\left(2 + 16s^2 \sin^2 \theta h\right)^2 - 4}$$

Certainly, here the worst case occurs for $\sin^2 \theta h = 1$:

$$x^2 = 1 + 8s^2 + \frac{1}{2}\sqrt{\left(2 + 16s^2\right)^2 - 4}$$

This, however, clearly has $x^2 = \left| e^{\lambda k} \right|^2 > 1$ for all real s.

So, I see that this procedure is unstable for all choices of s.

Kincaid 9.7.5) Investigate the stability of Euler's method:

$$v_{j,n+1} = v_{j,n} + \frac{\alpha k}{2h}\left(v_{j+1,n} - v_{j-1,n}\right)$$

for the equation:

$$\begin{aligned} u_t &= \alpha u_x \\ u(x,0) &= f(x) \end{aligned}$$

Using a Fourier argument, I take $v_{j,n} = e^{ij\theta h} e^{n\lambda k}$, with $i = \sqrt{-1}$.
Further, I define $s \equiv \dfrac{k\alpha}{2h}$.

$$e^{\lambda k} = 1 + s\left(e^{i\theta h} - e^{-i\theta h}\right)$$
$$e^{\lambda k} = 1 + 2is \sin \theta h$$

Multiplying each side by its complex conjugate,
$$\left| e^{\lambda k} \right|^2 = 1 + 2s^2 \sin^2 \theta h$$

However, I see that in the worst case:
$$\left| e^{\lambda k} \right|^2 = 1 + 2s^2 > 1$$

This indicates that then $\left| e^{\lambda k} \right| > 1$ for all non-zero choices of s, so that Euler's method is unstable for this equation.