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Numerical Analysis Homework 5

Kincaid 8.3.3) Prove that the Runge-Kutta formula:

$$x(t+h) = x(t) + \frac{1}{6}\left(F_1 + 2F_2 + 2F_3 + F_4\right)$$

$$F_1 = hf(t,x)$$

$$F_2 = hf\left(t + \frac{1}{2}h, x + \frac{1}{2}F_1\right)$$

$$F_3 = hf\left(t + \frac{1}{2}h, x + \frac{1}{2}F_2\right)$$

$$F_4 = hf\left(t + h, x + F_3\right)$$

is of order 4 in the special case that $f(t,x)$ is independent of x. Show that in this case the Runge-Kutta formula is equivalent to Simpson's Rule:

$$\int_a^b f(x)dx \approx \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

If f is independent of x, then I have:

$$F_1 = hf(t) \quad F_2 = F_3 = hf\left(t + \frac{1}{2}h\right) \quad F_4 = hf(t+h)$$

The result I expect to see comes from Taylor:

$$x(t+h) = x(t) + hf(t) + \frac{1}{2}h^2 f'(t) + \frac{1}{6}h^3 f''(t) + O\left(h^4\right)$$

In the special case of $f(t,x)$ independent of x:

$$f\left(t + \frac{1}{2}h\right) = f(t) + \frac{h}{2}f'(t) + \frac{h^2}{4}f''(t) + \ldots$$

$$f(t+h) = f(t) + hf'(t) + \frac{h^2}{2}f''(t) + \ldots$$

Making these substitutions in the Runge-Kutta formula, I have:

$$x(t+h) = x(t) + \frac{1}{6}\left(hf(t) + 4f\left(t + \frac{h}{2}\right) + hf(t+h)\right)$$

$$= x(t) + \frac{h}{6}f(t) + \frac{2h}{3}\left[f(t) + \frac{h}{2}f'(t) + \frac{h^2}{8}f''(t)\right] + \frac{h}{6}\left[f(t) + hf'(t) + \frac{h^2}{2}f''(t)\right] + O\left(h^4\right)$$

$$= x(t) + f(t) + \frac{1}{2}hf'(t) + \frac{1}{6}h^2 f''(t) + O\left(h^4\right)$$

The above agrees exactly with Taylor, as expected.

Then, I see that this is equivalent to Simpson's in the special case of $f(t,x)$ independent of x by simple substitution:

$$x(t+h)-x(t)=\frac{h}{6}\left(f(t)+4f\left(t+\frac{1}{2}h\right)+f(h)\right)$$

$$t+h\to b \quad t\to a$$

$$\int_a^b f(x)dx\approx\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

Kincaid 8.3.4) Derive the modified Euler's method:

$$x(t+h)=x(t)+hf\left(t+\frac{1}{2}h,x(t)+\frac{1}{2}hf(t,x(t))\right)$$

by performing Richardson's extrapolation on Euler's method using step sizes h and $\frac{h}{2}$. Assume the error term is Ch^2.

Euler's Method is given by $x(t+h)=x(t)+hf(t,x(t))$.

The strategy at hand in this modified formula is this: I'd like to use, instead of the derivative at this time and position, use the derivative at a time halfway to my target time and a position halfway to my target position for a better approximation.

Then:

$$x'(t)=f(t,x(t))$$

Starting with two steps leading to the desired formula, I have:

$$(A): x(t+h)=x(t)+hx'(t)+O(h^2)$$

$$(B): x(t+h)=x\left(t+\frac{h}{2}\right)+\frac{h}{2}x'\left(t+\frac{h}{2}\right)+O(h^2)$$

However, formula A:

$$(C): x\left(t+\frac{h}{2}\right)=x(t)+\frac{h}{2}x'(t)+O(h^2)$$

Then, substituting formula C into formula B:

$$(D): x(t+h)=x(t)+\frac{h}{2}x'(t)+\frac{h}{2}x'\left(t+\frac{h}{2}\right)+O(h^2)$$

Now, in another Taylor expansion:

$$(E): x'(t) = x'\left(t+\frac{h}{2}\right) - \frac{h}{2}x''\left(t+\frac{h}{2}\right) + O(h^2)$$

Substituing this into formula D:

$$(F): x(t+h) = x(t) + hx'\left(t+\frac{h}{2}\right) + O(h^2)$$

Taking

$$(H): x'\left(t+\frac{h}{2}\right) = f\left(t+\frac{h}{2}, x\left(t+\frac{h}{2}\right)\right)$$

and substituting formula C, then, I have:

$$(I): x'\left(t+\frac{h}{2}\right) = f\left(t+\frac{h}{2}, x(t) + \frac{h}{2}x'(t) + O(h^2)\right)$$

Substituting formula I into formula F, then, using $x'(t) = f(t, x(t))$ and dropping terms of order h^2, I have:

$$x(t+h) \approx x(t) + hf\left(t+\frac{1}{2}h, x(t) + \frac{1}{2}hf(t, x(t))\right)$$

Just as expected. Thus, I have used an analog of Richardson’s extrapolation to advance to this version of Euler’s method.

Kincaid 8.3.5) Derive the third-order Runge-Kutta formula

$$x(t+h) = x(t) + \frac{1}{9}(2F_1 + 3F_2 + 4F_3)$$

where

$$F_1 = hf(t, x)$$

$$F_2 = hf\left(t+\frac{1}{2}h, x+\frac{1}{2}F_1\right)$$

$$F_3 = hf\left(t+\frac{3}{4}h, x+\frac{3}{4}F_2\right)$$

Show that it agrees with the Taylor-series method of order 3 for the differential equation $x' = x + t$.

Let $x' = f(t, x)$.

Then,

$$x' = f(t,x)$$
$$x'' = f_t + f_x x' = f_t + f_x f$$
$$x^{(3)} = f_{tt} + f_{xt} f + f_x f_t + \left(f_{tx} + f_{xx} f + f_x f_x\right) f = f_{tt} + 2 f_{xt} f + f_x f_t + f_{xx} f^2 + f_x^2 f$$

Now, from Taylor I have:

$$x(t+h) = x + hf + \frac{1}{2}h^2\left(f_t + f_x f\right) + \frac{1}{6}h^3\left(f_{tt} + 2 f_{xt} f + f_x f_t + f_{xx} f^2 + f_x^2 f\right) + O\left(h^4\right)$$

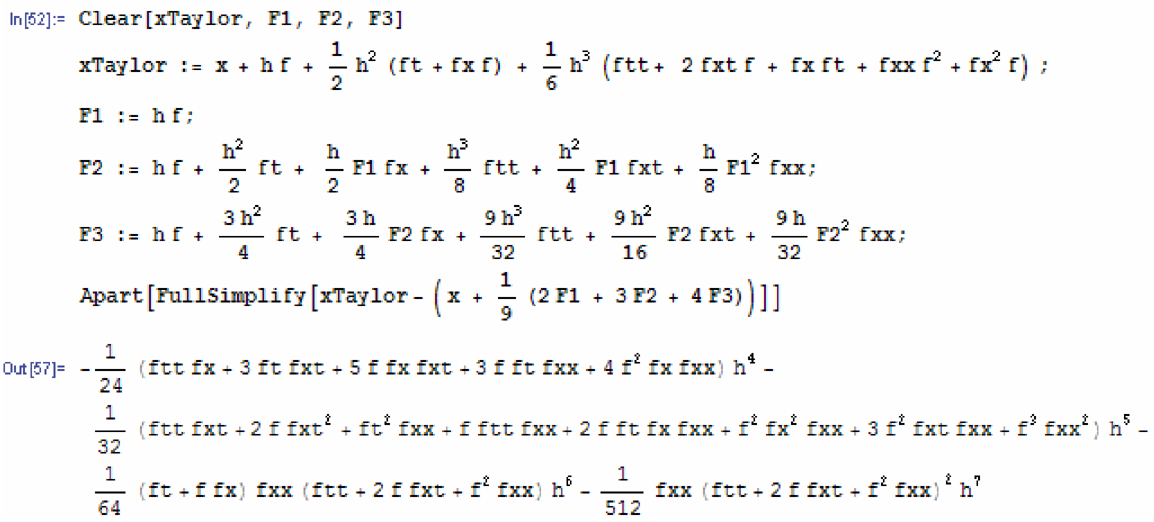
I will need to match these partial derivatives using the functions F. Expanding the chosen functions (these are arbitrary) out to order h^4:

$$F_1 = hf(t,x)$$

$$F_2 = hf\left(t + \frac{1}{2}h, x + \frac{1}{2}F_1\right) = hf(t,x) + \frac{h^2}{2} f_t + \frac{h}{2} F_1 f_x + \frac{h^3}{8} f_{tt} + \frac{h^2}{4} F_1 f_{xt} + \frac{h}{8} F_1^2 f_{xx} + O\left(h^4\right)$$
$$= hf(t,x) + \frac{h^2}{2} f_t + \frac{h^2}{2} ff_x + \frac{h^3}{8} f_{tt} + \frac{h^3}{4} ff_{xt} + \frac{h^3}{8} f^2 f_{xx} + O\left(h^4\right)$$

$$F_3 = hf\left(t + \frac{3}{4}h, x + \frac{3}{4}F_2\right)$$
$$= hf(t,x) + \frac{3h^2}{4} f_t + \frac{3h}{4} F_2 f_x + \frac{9h^3}{32} f_{tt} + \frac{9h^2}{16} F_2 f_{xt} + \frac{9h}{8} F_2^2 f_{xx} + O\left(h^4\right)$$
$$= hf + \frac{3h^2}{4} f_t + \frac{3h}{4}\left[hf + \frac{h^2}{2} f_t + \frac{h^2}{2} ff_x\right] f_x + \frac{9h^3}{32} f_{tt} + \frac{9h^3}{16} ff_{xt} + \frac{9h^3}{32} f^2 f_{xx} + O\left(h^4\right)$$

The final step is to add a combination of these to the effect of agreeing with the Taylor series for $x(t+h)$ up to order 3. As is easily seen by the complexity of the above formula, such a solution is best left for a computer algebra system. I verify this solution in Mathematica:



In[52]:= Clear[xTaylor, F1, F2, F3]

$$\mathtt{xTaylor} := x + h\,f + \frac{1}{2}h^2\,(\mathtt{ft} + \mathtt{fx}\,f) + \frac{1}{6}h^3\left(\mathtt{ftt} + 2\,\mathtt{fxt}\,f + \mathtt{fx}\,\mathtt{ft} + \mathtt{fxx}\,f^2 + \mathtt{fx}^2\,f\right);$$

$$\mathtt{F1} := h\,f;$$

$$\mathtt{F2} := h\,f + \frac{h^2}{2}\,\mathtt{ft} + \frac{h}{2}\,\mathtt{F1}\,\mathtt{fx} + \frac{h^3}{8}\,\mathtt{ftt} + \frac{h^2}{4}\,\mathtt{F1}\,\mathtt{fxt} + \frac{h}{8}\,\mathtt{F1}^2\,\mathtt{fxx};$$

$$\mathtt{F3} := h\,f + \frac{3\,h^2}{4}\,\mathtt{ft} + \frac{3\,h}{4}\,\mathtt{F2}\,\mathtt{fx} + \frac{9\,h^3}{32}\,\mathtt{ftt} + \frac{9\,h^2}{16}\,\mathtt{F2}\,\mathtt{fxt} + \frac{9\,h}{32}\,\mathtt{F2}^2\,\mathtt{fxx};$$

$$\mathtt{Apart}\left[\mathtt{FullSimplify}\left[\mathtt{xTaylor} - \left(x + \frac{1}{9}\,(2\,\mathtt{F1} + 3\,\mathtt{F2} + 4\,\mathtt{F3})\right)\right]\right]$$

Out[57]=

$$-\frac{1}{24}\,(\mathtt{ftt}\,\mathtt{fx} + 3\,\mathtt{ft}\,\mathtt{fxt} + 5\,f\,\mathtt{fx}\,\mathtt{fxt} + 3\,f\,\mathtt{ft}\,\mathtt{fxx} + 4\,f^2\,\mathtt{fx}\,\mathtt{fxx})\,h^4 -$$
$$\frac{1}{32}\,(\mathtt{ftt}\,\mathtt{fxt} + 2\,f\,\mathtt{fxt}^2 + \mathtt{ft}^2\,\mathtt{fxx} + f\,\mathtt{ftt}\,\mathtt{fxx} + 2\,f\,\mathtt{ft}\,\mathtt{fx}\,\mathtt{fxx} + f^2\,\mathtt{fx}^2\,\mathtt{fxx} + 3\,f^2\,\mathtt{fxt}\,\mathtt{fxx} + f^3\,\mathtt{fxx}^2)\,h^5 -$$
$$\frac{1}{64}\,(\mathtt{ft} + f\,\mathtt{fx})\,\mathtt{fxx}\,(\mathtt{ftt} + 2\,f\,\mathtt{fxt} + f^2\,\mathtt{fxx})\,h^6 - \frac{1}{512}\,\mathtt{fxx}\,(\mathtt{ftt} + 2\,f\,\mathtt{fxt} + f^2\,\mathtt{fxx})^2\,h^7$$

Clearly, the first remaining terms are of order h^4.

Kincaid 8.3.6) Prove that when the fourth-order Runge-Kutta method is applied to the problem $x' = \lambda x$, the formula for advancing this solution will be

$$x(t+h) = \left[1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4\right]x(t)$$

The classical fourth-order Runge-Kutta method is given by:

$$x(t+h) = x(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$
$$F_1 = hf(t, x)$$
$$F_2 = hf\left(t + \frac{1}{2}h, x + \frac{1}{2}F_1\right)$$
$$F_3 = hf\left(t + \frac{1}{2}h, x + \frac{1}{2}F_2\right)$$
$$F_4 = hf(t + h, x + F_3)$$

Under this substitution, then:

$$F_1 = h\lambda x$$

Further, simple substitution gives:

$$F_2 = hf\left(t+\frac{1}{2}h, x+\frac{1}{2}h\lambda x\right) = h\lambda\left(x+\frac{1}{2}h\lambda x\right)$$

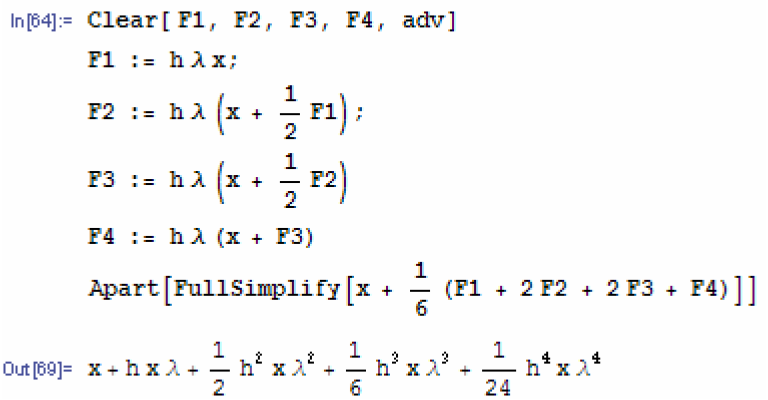
$$F_3 = hf\left(t+\frac{1}{2}h, x+\frac{1}{2}F_2\right) = h\lambda\left(x+h\lambda\left(x+\frac{1}{2}h\lambda x\right)\right)$$

$$F_4 = hf\left(t+h, x+F_3\right) = h\lambda\left(x+h\lambda\left(x+h\lambda\left(x+\frac{1}{2}h\lambda x\right)\right)\right)$$

Substituting these into the formula:

$$x(t+h) = x(t) + \frac{1}{6}\left(F_1 + 2F_2 + 2F_3 + F_4\right)$$

gives:



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In[64]:= Clear[F1, F2, F3, F4, adv]
         F1 := h λ x;
         F2 := h λ (x + 1/2 F1);
         F3 := h λ (x + 1/2 F2)
         F4 := h λ (x + F3)
         Apart[FullSimplify[x + 1/6 (F1 + 2 F2 + 2 F3 + F4)]]

Out[69]= x + h x λ + 1/2 h^2 x λ^2 + 1/6 h^3 x λ^3 + 1/24 h^4 x λ^4
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This clearly agrees with the expected formula.

Kincaid 8.6.6) Write an autonomous system of first-order equations equivalent to:

$$x''' - \left[\sin x'' + e^t x'\right]^2 + \cos x = 0$$
$$x(0) = 3 \quad x'(0) = 4 \quad x''(0) = 5$$

Take:

$$x_0 = e^t \quad x_1 = x \quad x_2 = x' \quad x_3 = x''$$

$e.g.$

$$x_0' = x_0 \quad x_1' = x_2 \quad x_2' = x_3$$
$$x_3' = \left[\sin x_3 + x_0 x_2\right]^2 + \cos x_1$$