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 Numerical Analysis Homework 4

**Kincaid 7.1 #3)** Let a numerical process be described by  $L = \varphi(h) + \sum_{j=1}^{\infty} a_j h^j$ .

**Explain how Richardson extrapolation will work in this case. Prove an analogue of Theorem 1 in this section on Richardson Extrapolation for this case.**

Consider two values of  $L = L_k(x)$  and  $L = L_k\left(\frac{x}{2}\right)$ , defined as:

$$L = L_k(x) = \varphi(x) + \sum_{j=k}^{\infty} a_j x^j$$

$$L = L_k\left(\frac{x}{2}\right) = \varphi\left(\frac{x}{2}\right) + \sum_{j=k}^{\infty} a_j \left(\frac{x}{2}\right)^j$$

Then,

$$(1 - 2^k)L = L_k(x) - 2^k L_k\left(\frac{x}{2}\right) = O(x^{k+1})$$

$$L + O(x^{k+1}) = \frac{2^k}{2^k - 1} L_k\left(\frac{x}{2}\right) - \frac{1}{2^k - 1} L_k(x)$$

This justifies this procedure:

First, select a convenient value of  $h$  and compute the  $M + 1$  numbers:

$$D(n,0) = \varphi\left(\frac{h}{2^n}\right) \quad (0 \leq n \leq M).$$

Compute each additional quantity by:

$$D(n,k) = \frac{2^k}{2^k - 1} D(n,k-1) - \frac{1}{2^k - 1} D(n-1,k-1) \quad (0 \leq [k \in \mathbb{Z}] \leq M) \quad (k \leq [n \in \mathbb{Z}] \leq M)$$

So that now  $D(n,k-1) = L + O(h^{2n})$ .

Namely, I guess that

$$D(n,k-1) = L + \sum_{j=k}^{\infty} A_{j,k} \left(\frac{h}{2^n}\right)^j.$$

To show this, I see that clearly:

$$D(n,0) = \varphi\left(\frac{h}{2^n}\right) = L - \sum_{j=1}^{\infty} a_j \left(\frac{h}{2^n}\right)^j$$

So that I let  $A_{j,1} = -a_j$ .

Proceeding by induction, I have that:

$$D(n,k) = \frac{2^k}{2^k - 1} \left[ L - \sum_{j=k}^{\infty} A_{j,k} \left(\frac{h}{2^n}\right)^j \right] - \frac{1}{2^k - 1} \left[ L - \sum_{j=k}^{\infty} A_{j,k} \left(\frac{h}{2^{n-1}}\right)^j \right]$$

$$D(n,k) = L + \sum_{j=k}^{\infty} A_{j,k} \left[ \frac{2^k - 2^j}{2^k - 1} \right] \left(\frac{h}{2^n}\right)^j$$

Clearly, then,

$$A_{j,k+1} = A_{j,k} \left[ \frac{2^k - 2^j}{2^k - 1} \right] \text{ and } D(n,k-1) = L + \sum_{j=k}^{\infty} A_{j,k} \left(\frac{h}{2^n}\right)^j.$$

**Kincaid 7.1 #6) Derive the following two formulas for approximating derivatives and show that they are both  $O(h^4)$  by establishing their error terms:**

$$f'(x) \approx \frac{1}{12h} [-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)]$$

$$f''(x) \approx \frac{1}{12h^2} [-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)]$$

First, note that:

$$f(x-2h) = f(x) - 2hf'(x) + \frac{(2h)^2}{2} f''(x) - \frac{(2h)^3}{6} f'''(x) + \frac{(2h)^4}{24} f^{(4)}(x) - \frac{(2h)^5}{120} f^{(5)}(\xi)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f^{(4)}(x) - \frac{h^5}{120} f^{(5)}(\xi)$$

$$f(x) = f(x)$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f^{(4)}(x) + \frac{h^5}{120} f^{(5)}(\xi)$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2} f''(x) + \frac{(2h)^3}{6} f'''(x) + \frac{(2h)^4}{24} f^{(4)}(x) + \frac{(2h)^5}{120} f^{(5)}(\xi)$$

Finding these formulae is a simple matter of selecting the derivative of interest and eliminating the unwanted derivatives in favor of the functions on the left-hand side.

Doing so using Solve in Mathematica, one is left with:

$$f'(x) = \frac{1}{12h} [-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)] + \frac{1}{30} h^4 f^{(5)}(\xi)$$

$$f''(x) = \frac{1}{12h^2} [-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)] + \frac{1}{90} h^4 f^{(6)}(\xi)$$

**Kincaid 7.1 #12) Show how to use Richardson extrapolation if**

$$L = \varphi(h) + a_1 h + a_3 h^3 + a_5 h^5 + \dots$$

I simply need to apply the formula from #3 once, then the one derived in the book for all other iterations:

First, select a convenient value of  $h$  and compute the  $M + 1$  numbers:

$$D(n,0) = \varphi\left(\frac{h}{2^n}\right) \quad (0 \leq n \leq M).$$

Compute the first quantity by:

$$D(n,1) = \frac{2^k}{2^k - 1} D(n,0) - \frac{1}{2^k - 1} D(n-1,0) \quad \text{with } (1 \leq [n \in Z] \leq M)$$

So that now  $D(n,0) = L + O(h)$ .

Compute each additional quantity by:

$$D(n,k) = \frac{4^k}{4^k - 1} D(n,k-1) - \frac{1}{4^k - 1} D(n-1,k-1) \quad (1 < [k \in Z] \leq M) \quad (k \leq [n \in Z] \leq M)$$

with  $D(n,k > 0) = L + O(h^{2^{k-1}})$

**Kincaid 7.1 #14) Using Taylor series, derive the error term for the approximation:**

$$f'(x) \approx \frac{1}{2h} [-3f(x) + 4f(x+h) - f(x+2h)]$$

Substituting this into Mathematica using the same series as in problem 7.1 #36 above, I see that:

$$f'(x) = \frac{1}{2h} [-3f(x) + 4f(x+h) - f(x+2h)] + \frac{1}{3} h^2 f^{(3)}(\xi)$$

**Kincaid 7.1 #15) Derive a numerical differentiation formula of order  $O(h^4)$  by applying Richardson extrapolation to:**

$$f'(x) = \frac{1}{2h} [f(x+h) - f(x-h)] - \frac{h^2}{6} f^{(3)}(x) - \frac{h^4}{120} f^{(5)}(x) - \dots$$

Give the error term of order  $O(h^4)$ .

Using the notation from Kincaid, p. 175,

$$L = \frac{1}{2h} [f(x+h) - f(x-h)] - \frac{h^2}{6} f^{(3)}(x) - \frac{h^4}{120} f^{(5)}(x) - \dots$$

$$\varphi(h) = \frac{1}{2h} [f(x+h) - f(x-h)]$$

$$a_1 = \frac{1}{6} f^{(3)}(x) \quad a_2 = \frac{1}{120} f^{(5)}(x)$$

In order to determine the new formula, I need  $D(2,1) = L + O(h^4)$ .

$$D(0,0) = \varphi(h)$$

$$D(1,0) = \varphi\left(\frac{h}{2}\right)$$

$$D(2,0) = \varphi\left(\frac{h}{4}\right)$$

$$\begin{aligned} D(2,1) &= \frac{4}{3} D(2,0) - \frac{1}{4} D(1,0) = \frac{4}{3} \varphi\left(\frac{h}{4}\right) - \frac{1}{4} \varphi\left(\frac{h}{2}\right) \\ &= \frac{8}{3h} \left( f\left(x + \frac{h}{4}\right) - f\left(x - \frac{h}{4}\right) \right) - \frac{1}{4h} \left( f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \right) = L + O(h^4) \end{aligned}$$

In particular, it is then clear from the pre-factors in the form above that I am adding:

$$err = \frac{16}{3} \left[ \left(\frac{h}{4}\right)^4 \frac{1}{120} f^{(5)}(\xi) \right] - \frac{1}{2} \left[ \left(\frac{h}{2}\right)^4 \frac{1}{120} f^{(5)}(\xi) \right] = -\frac{1}{96} \left[ \frac{h^4}{120} f^{(5)}(\xi) \right].$$

**Kincaid 7.2 #4) Verify that the following formula is exact for polynomials of degree**

$$\leq 4: \int_0^1 f(x) dx \approx \frac{1}{90} \left[ 7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1) \right].$$

In order to verify this, I try the formula for the polynomials  $1, x, x^2, x^3, x^4$ . If the formula holds for each of these, it will hold for all linear combinations thereof.

$$\int_0^1 1 dx = 1 = \frac{1}{90} [7 + 32 + 12 + 32 + 7]$$

$$\int_0^1 x dx = \frac{1}{2} = \frac{1}{90} \left[ 32 \left( \frac{1}{4} \right) + 12 \left( \frac{1}{2} \right) + 32 \left( \frac{3}{4} \right) + 7 \right]$$

$$\int_0^1 x^2 dx = \frac{1}{3} = \frac{1}{90} \left[ 32 \left( \frac{1}{4} \right)^2 + 12 \left( \frac{1}{2} \right)^2 + 32 \left( \frac{3}{4} \right)^2 + 7 \right]$$

$$\int_0^1 x^3 dx = \frac{1}{4} = \frac{1}{90} \left[ 32 \left( \frac{1}{4} \right)^3 + 12 \left( \frac{1}{2} \right)^3 + 32 \left( \frac{3}{4} \right)^3 + 7 \right]$$

$$\int_0^1 x^4 dx = \frac{1}{5} = \frac{1}{90} \left[ 32 \left( \frac{1}{4} \right)^4 + 12 \left( \frac{1}{2} \right)^4 + 32 \left( \frac{3}{4} \right)^4 + 7 \right]$$

Incidentally, the formula is also exact for polynomials of degree 5.

$$\int_0^1 x^5 dx = \frac{1}{6} = \frac{1}{90} \left[ 32 \left( \frac{1}{4} \right)^5 + 12 \left( \frac{1}{2} \right)^5 + 32 \left( \frac{3}{4} \right)^5 + 7 \right]$$

**Kincaid 7.2 #5) From the formula in the preceding problem, obtain a formula for  $\int_a^b f(x) dx$  that is exact for all polynomials of degree 4.**

Certainly, it is clear that  $\int_0^1 f(x) dx \approx \frac{1}{90} \left[ 7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1) \right]$  is exact for all polynomials of degree 4.

Now take  $\int_a^b f(x) dx$ . Define  $y = \frac{x-a}{b-a}$ . Now,  $dy = \frac{1}{b-a} dx$ .

Then,  $\int_a^b f(x) dx = (b-a) \int_0^1 f((b-a)y + a) dy$ .

However, note that  $f((b-a)y + a)$  is itself a fourth-degree polynomial if  $f(x)$  is a fourth-degree polynomial, and so the equation should still hold for  $g(y) = f((b-a)y + a)$ . Then, making this substitution,

$$\int_a^b f(x) dx \approx \frac{(b-a)}{90} \left[ 7f(a) + 32f\left(\frac{(b-a)}{4} + a\right) + 12f\left(\frac{(b-a)}{2} + a\right) + 32f\left(\frac{3}{4}(b-a) + a\right) + 7f(b) \right]$$

**Kincaid 7.2 #8) Find the formula  $\int_0^1 f(x)dx \approx A_0 f(0) + A_1 f(1)$  that is exact for all**

**functions of the form  $f(x) = ae^x + b \cos\left(\frac{\pi x}{2}\right)$ .**

$$\int_0^1 e^x dx = e - 1 = A_0 + eA_1$$

$$\int_0^1 \cos\left(\frac{\pi x}{2}\right) dx = \frac{2}{\pi} \sin\left(\frac{\pi}{2}\right) - \frac{2}{\pi} \sin(0) = \frac{2}{\pi} = A_0$$

Then,

$$A_0 + eA_1 = e - 1$$

$$A_0 = \frac{2}{\pi}$$

$$A_1 = \frac{e - 1 - \frac{2}{\pi}}{e}$$

So then,

$$\int_0^1 f(x) dx \approx \frac{2}{\pi} f(0) + \left(1 - e^{-1} - \frac{2}{\pi e}\right) f(1)$$

is exact for functions of the form  $f(x) = ae^x + b \cos\left(\frac{\pi x}{2}\right)$ .

**Kincaid 7.2 #12) Derive a formula for approximating  $\int_1^3 f(x) dx$  in terms of  $f(0), f(2), f(4)$ . It should be exact for all  $f$  in  $\Pi_2$ .**

$$\int_1^3 [f(x) \in \Pi_2] dx = Af(0) + Bf(2) + Cf(4)$$

Then,

$$\int_1^3 dx = 2 = A + B + C$$

$$\int_1^3 x dx = 4 = 2B + 4C$$

$$\int_1^3 x^2 dx = \frac{26}{3} = 4B + 16C$$

Solving the system

$$2 = A + B + C$$

$$4 = 2B + 4C$$

$$\frac{26}{3} = 4B + 16C$$

gives

$$A = \frac{1}{12} \quad B = \frac{11}{6} \quad C = \frac{1}{12}$$

$$\int_1^3 f(x) dx \approx \frac{1}{12} f(0) + \frac{11}{6} f(2) + \frac{1}{12} f(4)$$

**Kincaid 7.2 #13) Determine values for  $A$ ,  $B$  and  $C$  that make the formula**

$$\int_0^2 xf(x) dx \approx Af(0) + Bf(1) + Cf(2) \text{ exact for all polynomials of degree as high as}$$

**possible. What is the maximum degree?**

I will attempt to ensure that they work for polynomials of at least degree 2. It may be that they accidentally work for much higher ones:

$$\int_0^2 x dx = 2 = A + B + C$$

$$\int_0^2 x^2 dx = \frac{8}{3} = B + 2C$$

$$\int_0^2 x^3 dx = 4 = B + 4C$$

Solving the system

$$2 = A + B + C$$

$$\frac{8}{3} = B + 2C$$

$$4 = B + 4C$$

gives

$$A = 0 \quad B = \frac{4}{3} \quad C = \frac{2}{3}$$

$$\int_0^2 xf(x) dx \approx \frac{4}{3} f(1) + \frac{2}{3} f(2)$$

Checking higher orders, I get:

$$\int_0^2 x^4 dx = \frac{32}{5} \neq \frac{4}{3} + \frac{2}{3} \quad (8)$$

So, it appears that the maximum degree for which this formula is exact is degree 2.

**Kincaid 7.2 #20) Determine the integration rule for  $\int_a^b f(x)dx$  based on the Gaussian**

**quadrature rule  $\int_{-1}^1 f(x)dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$ .**

Take  $\int_a^b f(x)dx$ . Define  $y = \frac{2x - a - b}{b - a}$ . Now,  $dy = \frac{2}{b - a} dx$ .

Then,  $\int_a^b f(x)dx = \frac{(b - a)}{2} \int_{-1}^1 f\left(\frac{(b - a)}{2}y + \frac{a + b}{2}\right)dy$ .

However,  $g(y) = f\left(\frac{(b - a)}{2}y + \frac{a + b}{2}\right)$  is itself susceptible to Gaussian quadrature. Then,

$$\int_a^b f(x)dx = \frac{(b - a)}{2} \int_{-1}^1 g(y)dy \approx \frac{(b - a)}{2} \left[ f\left(\frac{a - b}{2\sqrt{3}} + \frac{a + b}{2}\right) + f\left(\frac{b - a}{2\sqrt{3}} + \frac{a + b}{2}\right) \right].$$

**Kincaid 7.2 #23) Prove that if the formula  $\int_{-1}^1 f(x)dx \approx \sum_{i=0}^n A_i f(x_i)$  ( $n$  even) is exact for all polynomials of degree  $n$  and if the nodes are symmetrically placed about the origin, then the formula is exact for all polynomials of degree  $n + 1$ .**

As in the derivation of these formulas, it is sufficient to show

that  $\int_{-1}^1 x^{n+1} dx = 0 = \sum_{i=0}^n A_i f(x_i)$  ( $n$  even).

Let the points be indexed from left to right, symmetrically about the origin, so that

$$x_1 = -x_n, x_2 = -x_{n-1}, \text{ etc.}$$

Now consider only the even polynomials. If this is exact for all polynomials of even degree  $n$ , then the terms with  $f(x_i)$  or  $f(x_{n-i})$  each give the same contribution with the same coefficient.

Considering only odd polynomials, if this formula is exact for odd polynomials of degree  $n$ , then the terms with  $f(x_i)$  and  $f(x_{n-i})$  give opposite contributions with the same coefficient.

Note that, up to degree  $n$ , the solution for these coefficients is unique for this choice of points. Then, it is clear that one possible solution would have the coefficients  $A_i = A_{n-i}$ . This would automatically satisfy the odd polynomials, since then

$$\sum_{i=0}^n A_i f(x_i) \quad (n \text{ even}, f \text{ odd}) = \sum_{i=0}^{\frac{n}{2}} A_i f(x_i) + \sum_{i=\frac{n}{2}}^n A_i f(x_i) = \sum_{i=0}^{\frac{n}{2}} A_i f(x_i) - \sum_{i=0}^{\frac{n}{2}} A_i f(x_i) = 0$$

And the even integral over an odd function is zero.

Further, this leaves the exact number of degrees of freedom necessary to fix the remaining even polynomials. By uniqueness, then, I have shown that necessarily  $A_i = A_{n-i}$ .

In fact, then, such a formula must be exact for all odd polynomials, not just those of degree  $n+1$ , since

$$\sum_{i=0}^n A_i f(x_i) \quad (n \text{ even}, f \text{ odd}) = 0 = \int_{-1}^1 (f(x) \text{ odd}) dx$$

when the nodes are distributed symmetrically about the origin.

**Kincaid 7.3 #11) For what value of  $\alpha$  is this formula exact on  $\Pi_3$ ?**

$$\int_0^2 f(x) dx \approx f(\alpha) + f(2-\alpha)$$

$$\int_0^2 dx = 2 = 1 + 1 \text{ is automatically satisfied.}$$

$$\int_0^2 x dx = 2 = \alpha + 2 - \alpha \text{ is automatically satisfied.}$$

$$\int_0^2 x^2 dx = \frac{8}{3} = \alpha^2 + (2-\alpha)^2$$

Solving,

$$\frac{8}{3} = 4 - 4\alpha + 2\alpha^2 \rightarrow \alpha = 1 \pm \frac{1}{\sqrt{3}}$$

Now that I've reduced my only degree of freedom to a binary choice, I'd like to verify that it works for degree 3 polynomials (I note that the two choices result in the same formula).

$$\int_0^2 x^3 dx = 4 = \left(1 + \frac{1}{\sqrt{3}}\right)^3 + \left(1 - \frac{1}{\sqrt{3}}\right)^3.$$

So, the choice that  $\alpha = 1 \pm \frac{1}{\sqrt{3}}$  does indeed make this formula exact on  $\Pi_3$ .

**Kincaid 7.3 #15) Determine the coefficients  $A_0$ ,  $A_1$  and  $A_2$  that make the formula**

$$\int_0^2 f(x) dx \approx A_0 f(0) + A_1 f(1) + A_2 f(2) \text{ exact for all polynomials of degree 3.}$$

The polynomials of up to degree 2 are sufficient to fix the coefficients:

$$\int_0^2 dx = 2 = A_0 + A_1 + A_2$$

$$\int_0^2 x dx = 2 = A_1 + 2A_2$$

$$\int_0^2 x^2 dx = \frac{8}{3} = A_1 + 4A_2$$

Solving the system

$$2 = A_0 + A_1 + A_2$$

$$2 = A_1 + 2A_2$$

$$\frac{8}{3} = A_1 + 4A_2$$

I get:

$$A_0 = \frac{1}{3} \quad A_1 = \frac{4}{3} \quad A_2 = \frac{1}{3}$$

Verifying that this works for polynomials of degree 3, I have:

$$\int_0^2 x^3 dx = 4 = \frac{4}{3} + \frac{1}{3} \cdot 8.$$

**Kincaid 7.3 #17) If the formula  $\int_0^1 xf(x)dx = \sum_{i=0}^4 A_i f(x_i)$  is correct for all  $f \in \Pi_9$ , then**

**$x_0, x_1, x_2, x_3, x_4$  must be the zeroes of a fifth-degree polynomial  $q$  that has what properties?**

I then need, for each polynomial in  $\Pi_4$ , a fifth-degree polynomial  $q$  satisfying:

$$\int_0^1 xq(x)p(x)dx = 0$$

By the linearity of the integral operator, it is sufficient for this to be the case for this to work for each  $p(x) \in \bigcup_{n=0}^4 \{x^n\}$ .

**Kincaid 7.3 #21) Consider a numerical integration rule of the form**

$$\int_{-1}^1 f(x)dx \approx Af\left(-\sqrt{\frac{3}{5}}\right) + Bf(0) + Cf\left(\sqrt{\frac{3}{5}}\right).$$

**a) What is the linear system that must be solved in the method of undetermined coefficients for finding  $A$ ,  $B$  and  $C$ ? Solve for  $A$ ,  $B$  and  $C$ .**

Assuming that this is to be correct for polynomials of at least degree 2, then:

$$\int_{-1}^1 dx = 2 = A + B + C$$

$$\int_{-1}^1 x dx = 0 = -\sqrt{\frac{3}{5}}A + \sqrt{\frac{3}{5}}C$$

$$\int_{-1}^1 x^2 dx = \frac{2}{3} = \frac{3}{5}A + \frac{3}{5}C$$

So that the linear system is

$$2 = A + B + C$$

$$0 = -\sqrt{\frac{3}{5}}A + \sqrt{\frac{3}{5}}C$$

$$\frac{2}{3} = \frac{3}{5}A + \frac{3}{5}C$$

With solution:  $A = \frac{5}{9}$   $B = \frac{8}{9}$   $C = \frac{5}{9}$ .

b) What three integrals must be evaluated to determine  $A$ ,  $B$  and  $C$  in a Newton-Cotes formula? Solve for  $A$ ,  $B$  and  $C$ .

$$A_i = \int_{-1}^1 \ell_i(x) dx$$

$$\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} \quad (0 \leq i \leq n)$$

Then,

$$A = A_0 = \int_{-1}^1 \ell_0(x) dx = \int_{-1}^1 \left( \frac{x}{-\sqrt{3}} \right) \left( \frac{x - \sqrt{\frac{3}{5}}}{-\sqrt{\frac{3}{5}} - \sqrt{\frac{3}{5}}} \right) dx = \frac{5}{9}$$

$$B = A_1 = \int_{-1}^1 \ell_1(x) dx = \int_{-1}^1 \left( \frac{x + \sqrt{\frac{3}{5}}}{\sqrt{\frac{3}{5}}} \right) \left( \frac{x - \sqrt{\frac{3}{5}}}{-\sqrt{\frac{3}{5}}} \right) dx = \frac{8}{9}$$

$$C = A_2 = \int_{-1}^1 \ell_2(x) dx = \int_{-1}^1 \left( \frac{x + \sqrt{\frac{3}{5}}}{\sqrt{\frac{3}{5}} + \sqrt{\frac{3}{5}}} \right) \left( \frac{x}{\sqrt{\frac{3}{5}}} \right) dx = \frac{5}{9}$$

The results are just the same as by the method of undetermined coefficients.

**Kincaid 7.3 #25) Using the method of undetermined coefficients, find  $A$ ,  $B$  and  $C$  in the following rule, which should give exact results for polynomials of degree 2:**

$$\int_{-3h}^h f(x) dx \approx h[Af(0) + Bf(-h) + Cf(-2h)]$$

$$\int_{-3h}^h dx = 4h = h[A + B + C]$$

$$\int_{-3h}^h x dx = -4h^2 = h[-Bh - 2Ch]$$

$$\int_{-3h}^h x^2 dx = \frac{28}{3}h^3 = h[Bh^2 + 4Ch^2]$$

The resulting system of equations is then:

$$4 = A + B + C$$

$$-4 = -B - 2C$$

$$\frac{28}{3} = B + 4C$$

The solution is:  $A = \frac{8}{3}$   $B = -\frac{4}{3}$   $C = \frac{8}{3}$

**Kincaid 7.3 #31) Determine the nodes and weights for the Gaussian formula of the**

**form:**  $\int_{-1}^1 x^4 f(x) dx \approx A_0 f(x_0) + A_1 f(x_1)$ .

First, I require a polynomial  $q$  of degree 2 such that  $\int_{-1}^1 x^4 q(x) p(x) dx = 0$  for each  $p(x) \in \Pi_1$ .

Then,  $\int_{-1}^1 x^4 \cdot q(x) dx = 0$   $\int_{-1}^1 x^4 \cdot x \cdot q(x) dx = 0$ .

Now, letting  $q(x) = Ax^2 + Bx + C$ , I have:

$$\int_{-1}^1 x^4 \cdot [Ax^2 + Bx + C] dx = \frac{2}{7}A + \frac{2}{5}C = 0$$

$$\int_{-1}^1 x^4 \cdot x \cdot [Ax^2 + Bx + C] dx = \frac{2}{7}B = 0$$

Solving the system,

$$\frac{2}{7}A + \frac{2}{5}C = 0$$

$$\frac{2}{7}B = 0$$

I have:

$$B = 0 \quad C = -\frac{5}{7}A$$

My polynomial  $q$  is then  $q(x) = A \left[ x^2 - \frac{5}{7} \right]$  with roots  $x = \pm \sqrt{\frac{5}{7}}$ .

Now I simply need to solve:

$$\int_{-1}^1 x^4 dx = \frac{2}{5} = A_0 + A_1$$

$$\int_{-1}^1 x^4 \cdot x dx = 0 = -\sqrt{\frac{5}{7}}A_0 + \sqrt{\frac{5}{7}}A_1$$

With the system of equations:

$$\frac{2}{5} = A_0 + A_1$$

$$0 = -\sqrt{\frac{5}{7}}A_0 + \sqrt{\frac{5}{7}}A_1$$

Then I have:  $A_0 = A_1 = \frac{1}{5}$ .

Combining everything, I have:  $\int_{-1}^1 x^4 f(x) dx \approx \frac{1}{5} f\left(-\sqrt{\frac{5}{7}}\right) + \frac{1}{5} f\left(\sqrt{\frac{5}{7}}\right)$ .

**Kincaid 7.4 #1) Derive the equation:**

$$\int_{x_i}^{x_{i+1}} f(x) dx = \frac{h}{2} [f(x_i) + f(x_{i+1})] + \sum_{k=1}^{m-1} A_{2k} h^{2k} [f^{(2k-1)}(x_i) - f^{(2k-1)}(x_{i+1})] - A_{2m} h^{2m+1} f^{(2m)}(\xi_i)$$

**From the Euler-Maclaurin formula:**

$$\int_0^1 f(t) dt = \frac{1}{2} [f(0) + f(1)] + \sum_{k=1}^{m-1} A_{2k} [f^{(2k-1)}(0) - f^{(2k-1)}(1)] - A_{2m} f^{(2m)}(\xi_0)$$

Define the following:

$$g(t) = f(x_i + ht) \quad h = x_{i+1} - x_i \quad t = \frac{x - x_i}{h}$$

Applying the Euler-Maclaurin formula to  $g(t)$ , I have:

$$\int_0^1 g(t) dt = \frac{1}{2} [g(0) + g(1)] + \sum_{k=1}^{m-1} A_{2k} [g^{(2k-1)}(0) - g^{(2k-1)}(1)] - A_{2m} g^{(2m)}(\xi_0)$$

Or, substituting  $g(t) = f(x_i + ht)$ ,

$$\int_0^1 f(x_i + ht) dt = \frac{1}{2} [f(x_i) + f(x_i + h)] + \sum_{k=1}^{m-1} A_{2k} [f^{(2k-1)}(x_i) - f^{(2k-1)}(x_i + h)] - A_{2m} f^{(2m)}(x_i + h\xi_0)$$

Substituting  $h = x_{i+1} - x_i$ , I have:

$$\int_0^1 f(x_i + (x_{i+1} - x_i)t) dt = \frac{1}{2} [f(x_i) + f(x_{i+1})] + \sum_{k=1}^{m-1} A_{2k} [f_t^{(2k-1)}(x_i) - f_t^{(2k-1)}(x_{i+1})] - A_{2m} f_t^{(2m)}(x_i + (x_{i+1} - x_i)\xi_0)$$

Under the change of variable  $t = \frac{x - x_i}{h}$   $dt = \frac{1}{h} dx$   $\frac{\partial}{\partial t} = \frac{\partial x}{\partial t} \frac{\partial}{\partial x} = h \frac{\partial}{\partial x}$ :

$$\frac{1}{h} \int_{x_i}^{x_{i+1}} f(x) dx = \frac{1}{2} [f(x_i) + f(x_{i+1})] + \sum_{k=1}^{m-1} A_{2k} h^{2k-1} [f_x^{(2k-1)}(x_i) - f_x^{(2k-1)}(x_{i+1})] - A_{2m} h^{2m} f_x^{(2m)}(x_i + (x_{i+1} - x_i)\xi_0)$$

Or,

$$\int_{x_i}^{x_{i+1}} f(x) dx = \frac{h}{2} [f(x_i) + f(x_{i+1})] + \sum_{k=1}^{m-1} A_{2k} h^{2k} [f_x^{(2k-1)}(x_i) - f_x^{(2k-1)}(x_{i+1})] - A_{2m} h^{2m+1} f_x^{(2m)}(\xi_i)$$

Let  $\xi_i \equiv x_i + (x_{i+1} - x_i)\xi_0$

**Kincaid 7.4 #2) Derive the equation**

$$\int_a^b f(x) dx = \frac{h}{2} \sum_{i=0}^{2^n-1} [f(x_i) + f(x_{i+1})] + \sum_{k=1}^{m-1} A_{2k} h^{2k} [f^{(2k-1)}(a) - f^{(2k-1)}(b)] - A_{2m} (b-a) h^{2m+1} f^{(2m)}(\xi)$$

**From the equation:**

$$\int_{x_i}^{x_{i+1}} f(x) dx = \frac{h}{2} [f(x_i) + f(x_{i+1})] + \sum_{k=1}^{m-1} A_{2k} h^{2k} [f^{(2k-1)}(x_i) - f^{(2k-1)}(x_{i+1})] - A_{2m} h^{2m+1} f^{(2m)}(\xi_i).$$

**In particular, justify the conversion from**

$$h^{2m+1} \sum_{i=0}^{2^n-1} f^{(2m)}(\xi_i) \text{ to } A_{2m} (b-a) h^{2m+1} f^{(2m)}(\xi).$$

**Start with:**

$$\int_{x_i}^{x_{i+1}} f(x) dx = \frac{h}{2} [f(x_i) + f(x_{i+1})] + \sum_{k=1}^{m-1} A_{2k} h^{2k} [f^{(2k-1)}(x_i) - f^{(2k-1)}(x_{i+1})] - A_{2m} h^{2m+1} f^{(2m)}(\xi_i).$$

Taking the operator  $\sum_{i=0}^{2^n-1}$  on both sides, I let  $x_i = a + ih$  for  $0 \leq i \leq 2^n$  and  $h = \frac{b-a}{2^n}$ . In this case, then, the integrals on the left-hand side are arranged with their limits touching and can be merged into one integral. Further, in the second term on the right-hand side, I notice that in the resulting sum, each term  $f^{(2k-1)}(x_i)$  with  $0 < i < 2^n$  cancels with the contribution from another element of the sum, leaving only the first and last elements. Now I have:

$$\int_a^b f(x)dx = \frac{h}{2} \sum_{i=0}^{2^n-1} [f(x_i) + f(x_{i+1})] + \sum_{k=1}^{m-1} A_{2k} h^{2k} [f^{(2k-1)}(a) - f^{(2k-1)}(b)] - A_{2m} h^{2m+1} \sum_{i=0}^{2^n-1} f^{(2m)}(\xi_i)$$

In the error term, then, I take:

$$\begin{aligned} & A_{2m} h^{2m+1} \sum_{i=0}^{2^n-1} f^{(2m)}(\xi_i) \\ &= (b-a) A_{2m} h^{2m} \frac{\sum_{i=0}^{2^n-1} f^{(2m)}(\xi_i)}{2^n} \end{aligned}$$

Now I see that the contribution from the derivatives have become the average of the contributions from each derivative. Recall that the justification for this term is that some point must exist in the neighborhood of the Taylor series that the function is taken about which would make a truncated Taylor series exactly correct. This justifies that some point  $\xi$  must exist that is the average of all of these (by the mean-value theorem) which would correct the aggregate of each region. Then, I have the expected contribution

$$\frac{\sum_{i=0}^{2^n-1} f^{(2m)}(\xi_i)}{2^n} = f^{(2m)}(\xi)$$

And

$$\begin{aligned} \int_a^b f(x)dx &= \frac{h}{2} \sum_{i=0}^{2^n-1} [f(x_i) + f(x_{i+1})] + \sum_{k=1}^{m-1} A_{2k} h^{2k} [f^{(2k-1)}(a) - f^{(2k-1)}(b)] \\ &\quad - A_{2m} (b-a) h^{2m+1} f^{(2m)}(\xi) \end{aligned}$$

As expected.