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Numerical Analysis Homework 3

Kincaid 4.4 #1) Show that the norms $\|x\|_\infty, \|x\|_2, \|x\|_1$ satisfy the postulates (1), (2) and (3) for norms.

$\|x\|_\infty = \max_{1\le i\le n}|x_i|$
$(1)\quad \|x\| > 0 \quad if \quad x \in V \ne \vec{0}$

Clearly, if any nonzero element x^* of x exists, $|x^*| > 0$ and $\|x\|_\infty = \max_{1\le i\le n}|x_i| \ge x^* > 0$.

$(2)\quad \|\lambda x\| = |\lambda|\|x\| \quad if \quad x \in V, \lambda \in R$

$\|\lambda x\|_\infty = \max_{1\le i\le n}|\lambda x_i| = \max_{1\le i\le n}|\lambda||x_i| = |\lambda|\max_{1\le i\le n}|x_i| = |\lambda|\|x\|_\infty$

$(3)\quad \|x+y\| \le \|x\| + \|y\| \quad if \quad x, y \in V$

Using the triangle inequality for Real numbers,

$\|x+y\|_\infty = \max_{1\le i\le n}|x_i + y_i| \le \max_{1\le i\le n}(|x_i| + |y_i|)$
$\max_{1\le i\le n}(|x_i| + |y_i|) = \|x\|_\infty + \|y\|_\infty$
$\|x+y\|_\infty \le \max_{1\le i\le n}(|x_i| + |y_i|)$

$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$
$\|x\|_2^2 = \sum_{i=1}^{n} x_i^2$
$(1)\quad \|x\| > 0 \quad if \quad x \in V \ne \vec{0}$

Clearly, for all real numbers $r^2 \ge 0$. If any nonzero element x^* of x exists, $x^{*2} > 0$ and $\|x\|_2 \ge x^{*2} > 0$.

$(2)\quad \|\lambda x\| = |\lambda|\|x\| \quad if \quad x \in V, \lambda \in R$

$\|\lambda x\|_2 = \sqrt{\sum_{i=1}^{n}(\lambda x_i)^2} = |\lambda|\sqrt{\sum_{i=1}^{n} x_i^2} = |\lambda|\|x\|_2$

(3) $\|x+y\| \le \|x\| + \|y\| \quad if \quad x, y \in V$

$$\|x+y\|_2^2 = \sum_{i=1}^{n} (x_i + y_i)^2$$
$$\sum_{i=1}^{n} (x_i + y_i)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + 2\sum_{i=1}^{n} x_i y_i$$
$$(\|x\|_2 + \|y\|_2)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + 2\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}$$
$$\|x+y\|_2^2 \overset{?}{\le} (\|x\|_2 + \|y\|_2)^2$$
$$2\sum_{i=1}^{n} x_i y_i \overset{?}{\le} 2\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}$$
$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \overset{?}{\le} \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$$
$$* \quad \langle x, y \rangle^2 \overset{?}{\le} \|x\|_2^2 \|y\|_2^2$$

Rigorously, certainly $\|x - \alpha y\|^2 \ge 0$, indicating that then

$$\|x-\alpha y\|^2 = \langle x-\alpha y, x-\alpha y\rangle = \|x\|^2 + \alpha^2\|y\|^2 - 2\alpha\|x\|\|y\|$$
$$\langle x-\alpha y, x-\alpha y\rangle \ge 0$$
$$\therefore \|x\|^2 + \alpha^2\|y\|^2 - 2\alpha\langle x, y\rangle \ge 0$$
$$if \quad A\alpha^2 + B\alpha + C \ge 0,$$
$$B^2 - 4AC \le 0$$
$$B^2 - 4AC = 4\langle x, y\rangle^2 - 4\|x\|^2\|y\|^2$$
$$\therefore \langle x, y\rangle^2 \le \|x\|^2\|y\|^2$$

This verifies equation * above and verifies proposition (3) for the 2-norm.

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

(1) $\|x\| > 0 \quad if \quad x \in V \ne \vec{0}$

Clearly, if any nonzero element x^* of x exists, $|x^*| > 0$ and $\|x\|_1 = \sum_{i=1}^{n} |x_i| \ge x^* > 0$.

(2) $\|\lambda x\| = |\lambda|\|x\| \quad if \quad x \in V, \lambda \in R$

$$\|\lambda x\|_1 = \sum_{i=1}^{n} |\lambda x_i| = |\lambda| \sum_{i=1}^{n} |x_i| = |\lambda| \|x\|_1$$

$$(3) \quad \|x+y\| \le \|x\| + \|y\| \quad if \quad x, y \in V$$

Using the triangle inequality for Real numbers,

$$\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i|$$
$$|x_i + y_i| \le |x_i| + |y_i|$$
$$\therefore \|x+y\|_1 \le \sum_{i=1}^{n} |x_i| + |y_i|$$

Kincaid 4.4 #2) Show that $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$ for all $x \in R^n$ and that equalities can occur even for nonzero vectors.

Then $\|x\|_\infty^{\ 2} \le \|x\|_2^{\ 2} \le \|x\|_1^{\ 2}$

First consider $\|x\|_\infty^{\ 2} \le \|x\|_2^{\ 2}$.

Note that $|r| = \sqrt{r^2} \ge 0$ for any real number r. Now consider the largest element in absolute value x^* of x. Then, $\|x\|_\infty^{\ 2} = |x^*|^2 = x^{*2}$. However, in the norm $\|x\|_2^{\ 2} = \sum_{i=1}^{n} x_i^2 = x^{*2} + \sum_{i=1, i\ne *}^{n} x_i^2 = \|x\|_\infty^{\ 2} + \sum_{i=1, i\ne *}^{n} x_i^2$, I see that the value of this norm is at least as large as $\|x\|_\infty$, and is equal when the vector x whenever x has exactly one nonzero element. Thus, $\|x\|_\infty^{\ 2} \le \|x\|_2^{\ 2}$ and $\|x\|_\infty \le \|x\|_2$

Next consider $\|x\|_2^{\ 2} \le \|x\|_1^{\ 2}$. Then, I see that

$$\|x\|_2^{\ 2} = \sum_{i=1}^{n} x_i^2$$
$$\|x\|_1^{\ 2} = \sum_{i,j=1}^{n} |x_i||x_j| = \sum_{i=1}^{n} |x_i|^2 + \sum_{i=1, j=1, j\ne i}^{n} |x_i||x_j| = \|x\|_2^{\ 2} + \sum_{i=1, j=1, j\ne i}^{n} |x_i||x_j|$$

So that clearly $\|x\|_1^{\ 2}$ is at least as large as $\|x\|_2^{\ 2}$, plus cross terms. Thus, $\|x\|_2^{\ 2} \le \|x\|_1^{\ 2}$ and $\|x\|_2 \le \|x\|_1$, with equality whenever x has exactly one nonzero element.

Kincaid 4.4 #3) Show that $\|x\|_1 \le n\|x\|_\infty$ and $\|x\|_2 \le \sqrt{n}\|x\|_\infty$ for all $x \in R^n$.

In $\|x\|_\infty = \max_{1\le i\le n}|x_i|$, replace $x_i = |x_{\max}|$ so that $\|x\|_\infty = |x_{\max}|$.

Then,

$$\|x\|_1 = \sum_{i=1}^{n}|x_i| \le \sum_{i=1}^{n}|x_{\max}| = n|x_{\max}| = n\|x\|_\infty$$

$$\therefore \|x\|_1 \le n|x_\infty|$$

Further,

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \le \sqrt{\sum_{i=1}^{n}|x_{\max}|^2} = \sqrt{n|x_{\max}|^2} = \sqrt{n}|x_{\max}| = \sqrt{n}|x_{\max}|$$

$$\therefore \|x\|_2 \le \sqrt{n}\|x\|_\infty$$

Kincaid 4.4 #8) Define $\|A\| = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|$. Show that this is a matrix norm (that is, a norm on the linear space of all $n \times n$ matrices. Show that it is not subordinate to any vector norm. Does it conform to equation (9) and inequality (10)?

$$\|A\| = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|$$

$$(1) \quad \|A\| > 0 \quad if \quad A \ne 0$$

Clearly, if any nonzero element a^* of A exists, $|a^*| > 0$ and $\|A\| = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}| > 0$.

$$(2) \quad \|\lambda A\| = |\lambda|\|A\| \quad if \quad \lambda \in R$$

$$\|\lambda A\| = \sum_{i=1}^{n}\sum_{j=1}^{n}|\lambda a_{ij}| = |\lambda|\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}| = |\lambda|\|A\|$$

$$(3) \quad \|A+B\| \le \|A\| + \|B\|$$

Using the triangle inequality for Real numbers,

$$\|A+B\| = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij} + b_{ij}\right| \le \sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right| + \left|b_{ij}\right|$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right| + \left|b_{ij}\right| = \|A\| + \|B\|$$

$$\therefore \|A+B\| \le \|A\| + \|B\|$$

This norm cannot be subordinate to any vector norm. Any norm subordinate to a vector norm has the property $\|I\| = 1$ [equation (9)]; in this case, $\|I\| = n$, where I is the $n \times n$ identity. Considering [equation (10)], which is $\|AB\| \le \|A\|\|B\|$, I have:

$$\|AB\| = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\sum_{k=1}^{n} a_{ik} b_{kj}\right| \le \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left|a_{ik} b_{kj}\right| = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left|a_{ik}\right|\left|b_{kj}\right| \le \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\sum_{k=1}^{n}\left|a_{ik}\right|\right]\left[\sum_{l=1}^{n}\left|b_{lj}\right|\right]$$

$$= \sum_{i=1}^{n}\left[\sum_{k=1}^{n}\left|a_{ik}\right|\right]\sum_{j=1}^{n}\left[\sum_{l=1}^{n}\left|b_{lj}\right|\right] = \|A\|\|B\|$$

$$\therefore \|AB\| \le \|A\|\|B\|$$

Kincaid 4.4 #11) Show that for the vector norm $\|x\|_1$, the subordinate matrix norm is $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n}\left|a_{ij}\right|$.

By the definition of a subordinate matrix norm,
$\|A\| = \sup_{\|u\|_1 = 1}\left\{\|Au\|_1 : u \in R^n\right\}$

Then take $\|u\|_1 = 1$ to have exactly one nonzero entry, which itself is one. Specifically, define $u^{(j)}$ to have its j-th entry one and all other entries zero. This choice is justified since exactly one column must have the largest sum, and to allow it to "mix" with any other column with any other choice of form of u could only decrease the value in the supremum. Now the supremum becomes a simple max:
$\|A\| = \max_{1\le j\le n}\left\{\left\|Au^{(j)}\right\|_1\right\} = \max_{1\le j\le n} \sum_{i=1}^{n}\left|a_{ij}\right|$.

Kincaid 4.4 #17) Let A be an $n \times n$ matrix with inverse $C = (c_{ij})$. Show that in the solution of $Ax = b$, a perturbation of amount δ in b_j will cause a perturbation of $c_{ij}\delta$ in x_i.

$$Ax = b$$
$$CA = I$$
$$\therefore x = Cb$$

Then, if $x_i = c_{ik} b_k$ and entry j of b becomes $b_j + \delta$, $x_i = c_{ik} b_k = \sum_{k=1}^{n} c_{ik} b_k + c_{ij}\delta$.

Kincaid 4.4 #18) (continuation of #17) Prove that a perturbation of amount δ in a_{jk} will produce a perturbation of approximately $-c_{ij} x_k \delta$ in x_i.

Using $Ax = b$, this gives $a_{ij} x_j = b_i$ for $b_i = a_{ij} x_j + \delta x_k$.

However, note that in order to keep a *constant* b_i in light of this perturbation, I need to take $b_i = a_{ij} x_j - \delta x_k$. This is because I want to go backwards to obtain x from b, rather than forwards to obtain a perturbation in b from x.

Using $x = Cb$, this gives $x_j = c_{ji} b_i = c_{ji} a_{ij} x_j - c_{ji}\delta x_k = x_j - c_{ji}\delta x_k$, for a perturbation of $x_i = -c_{ij}\delta x_k$ (where I have switched dummy indices).

Kincaid 4.4 #33) For any $n \times n$ matrix, define $\|A\|_F = \sqrt{\left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2\right)}$ (This is called the Frobenius norm.) Is this a subordinate matrix norm? Answer the same question for $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$. Prove that these equations define norms on the vector space of all $n \times n$ matrices.

These are both norms on $n \times n$ matrices; the norm $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$ was shown to be a subordinate matrix norm in problem #11 above.

$\|A\|_F$ cannot possibly be a subordinate matrix norm, since $\|I\|_F = \sqrt{n} \neq I$ for an $n \times n$ matrix.

Note that $\|A\|_F$ can be immediately be seen as a norm on matrices by analogy to the vector 2-norm: simply consider the pair of indices taken to a single index $(i, j) \to ni + j$ and $\sum_{i=1}^{n}\sum_{j=1}^{n} \to \sum_{i=1}^{n^2}$. This satisfies all of the same norm properties, as shown for the vector 2-norm in problem 1.

$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$

$(1) \quad \|A\| > 0 \quad if \quad A \neq 0$

Clearly, if any nonzero element a^* of A exists, ${a^*}^2 > 0$ and $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| > 0$.

$(2)\quad \|\lambda A\| = |\lambda|\|A\| \quad if \quad \lambda \in R$

$$\|\lambda A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |\lambda a_{ij}| = \max_{1\le j\le n} |\lambda| \sum_{i=1}^{n} |a_{ij}| = |\lambda|\|A\|_1$$

$(3)\quad \|A+B\| \le \|A\| + \|B\|$

Using the triangle inequality for Real numbers,

$$\|A+B\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij} + b_{ij}| \le \max_{1\le j\le n} \left(\sum_{i=1}^{n} |a_{ij}| + \sum_{i=1}^{n} |b_{ij}| \right) \le \max_{1\le j\le n} \left(\sum_{i=1}^{n} |a_{ij}| \right) + \max_{1\le j\le n} \left(\sum_{i=1}^{n} |b_{ij}| \right)$$
$$= \|A\|_1 + \|B\|_1$$
$$\therefore \|A+B\|_1 \le \|A\|_1 + \|B\|_1$$

Kincaid 4.4 #49) Prove that if a square matrix A satisfies an inequality $\|Ax\| \ge \theta\|x\|$ for all x with $\theta > 0$, then A is nonsingular and $\|A^{-1}\| \le \theta^{-1}$. This is valid for any vector norm and its subordinate matrix norm.

$$\|x\| = \|AA^{-1}x\|$$
$$Because \quad Ax, x \in V \quad and \quad \|Ax\| \ge \theta\|x\| \quad \forall \quad x \in V$$
$$\|x\| = \|AA^{-1}x\| \ge \theta\|A^{-1}x\|$$
$$\|x\| \ge \theta\|A^{-1}x\|$$
$$\frac{1}{\theta}\|x\| \ge \|A^{-1}x\|$$
$$\therefore \sup_{\|x\|=1} \|A^{-1}x\| \le \frac{1}{\theta}$$

Empirically, I may think of it this way:

First, I note that a matrix is singular if and only if it has a non-empty null space. However, if $\|Ax\| \ge \theta\|x\|$ with $\theta > 0$, then $\|Ax\| = 0$ if and only if $x = \vec{0}$ and so A's null space consists of the empty set and so A is nonsingular.

Now, based on the form $\|Ax\| \geq \theta\|x\|$ with $\theta > 0$, it is clear that then each of A's eigenvalues λ_i must be positive with $\lambda_i \geq \theta$. Since a matrix's inverse must reverse each of these stretches along each eigenvector, then, the matrix inverse's eigenvalues must be given by λ_i^{-1} with the same eigenvectors. The largest of these eigenvalues in A^{-1} must then correspond to the smallest eigenvalue of the matrix A, $\lambda_{\min} \geq \theta$. Then it follows that in the inverse A^{-1} I have $\lambda_{\min}^{-1} \leq \theta^{-1}$, with the limiting case to this inequality corresponding to x along the eigenvector of A with the smallest eigenvalue.

Kincaid 4.5 #3) Prove that if $\|A\| < 1$ then $\left\|(I - A)^{-1}\right\| \geq \frac{1}{1+\|A\|}$.

Certainly, it is clear that $(1 - A)^{-1} = \sum_{i=1}^{\infty} A^k$ since

$$(1 - A)\sum_{k=0}^{\infty} A^k = \sum_{k=0}^{\infty} A^k - \sum_{k=0}^{\infty} A^{k+1} = I$$

Because $\|A\| < 1$.

$$\left\|(1 - A)^{-1}\right\| = \left\|\sum_{k=0}^{\infty} A^k\right\| \geq \sum_{k=0}^{\infty}\left(-\|A\|\right)^k = \frac{1}{1+\|A\|}$$

Kincaid 4.5 #19) Prove that if A is invertible then $\|Ax\| \geq \|x\|\left\|A^{-1}\right\|^{-1}$.

$$A^{-1}Ax = x$$

$$\left\|A^{-1}Ax\right\| = \|x\|$$

$$\left\|A^{-1}\right\|\|Ax\| \geq \|x\|$$

$$\|Ax\| \geq \|x\|\left\|A^{-1}\right\|^{-1}$$

Kincaid 4.6 #2) Prove that if A has this property (unit row diagonally dominant):

$$a_{ii} = 1 > \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|a_{ij}\right| \quad (1 \leq i \leq n)$$

then Richardson iteration is successful.

The Richardson iteration will then be successful if $\|I - A\| < 1$ for an arbitrary choice of norm. I choose $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \left|a_{ij}\right|$.

Now I have:

$$\|I - A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |\delta_{ii} - a_{ij}| = \max_{1\le i\le n} \sum_{\substack{j=1 \\ j\ne i}}^{n} |a_{ij}|$$

$$given \quad \sum_{\substack{j=1 \\ j\ne i}}^{n} |a_{ij}| < 1 \quad \therefore \quad \max_{1\le i\le n} \sum_{\substack{j=1 \\ j\ne i}}^{n} |a_{ij}| < 1 \quad \therefore \quad \|I - A\|_\infty < 1$$

Thus the Richardson iterations converge.

Kincaid 4.6 #12) Explain why, in the proof of Theorem 3, we cannot let $\varepsilon \to 0$ and conclude that A is similar to a diagonal matrix.

The argument in Theorem 3 is that since any square matrix is similar to an upper triangular matrix then by use of a matrix D and D^{-1} with $D = diag(\varepsilon, \varepsilon^2, \ldots)$ one can write $D^{-1}TD$ in order to make the entries of this triangular matrix arbitrarily small. The reasons why this cannot work are threefold:

If $\varepsilon = 0, D = [0]$, so that anything then multiplied by it is zero, and in fact if the zero matrix were allowed as a similarity transform then any matrix would be similar to any other.

If $\varepsilon = 0, D^{-1} = diag(\infty, \infty, \infty)$, a matrix with whom multiplication gives an undefined result.

Finally, supposing that these issues with D could be avoided, no diagonal matrix could ever take a triangular matrix to a diagonal matrix since multiplication with a diagonal matrix only rescales each row or column depending on left or right multiplication.

Kincaid 4.6 #33) Prove that if $\delta = \|I - Q^{-1}A\| < 1$ then $\|x^{(k)} - x\| \le \frac{\delta}{1-\delta}\|x^{(k)} - x^{(k+1)}\|$.

By the premise of these algorithms,

$$x^{(k)} = (I - Q^{-1}A)x^{(k-1)} + Q^{-1}b$$
$$x = (I - Q^{-1}A)x + Q^{-1}b$$

Then,

$$x^{(k)} - x - (x^{(k+1)} - x) = (I - Q^{-1}A)(x^{(k)} - x^{(k-1)})$$
$$\|x^{(k)} - x - (x^{(k+1)} - x)\| = \|(I - Q^{-1}A)(x^{(k)} - x^{(k-1)})\|$$
$$\|x^{(k)} - x - (x^{(k+1)} - x)\| \le \|I - Q^{-1}A\|\|x^{(k)} - x^{(k-1)}\|$$

By the triangle inequality,

$$\|x^{(k)} - x\| - \|x^{(k+1)} - x\| \le \delta\|x^{(k)} - x^{(k-1)}\|$$

However, as seen by the convergence of this procedure,
$\|x^{(k+1)} - x\| \le \|I - Q^{-1}A\|\|x^{(k)} - x\|$

Therefore,
$-\|x^{(k+1)} - x\| \ge -\|I - Q^{-1}A\|\|x^{(k)} - x\|$
and
$\|x^{(k)} - x\| - \delta\|x^{(k)} - x\| \le \delta\|x^{(k)} - x^{(k-1)}\|$
$\|x^{(k)} - x\| \le \dfrac{\delta}{1-\delta}\|x^{(k)} - x^{(k-1)}\|$
QED.

Kincaid 4.6 #39) Prove that if A is nonsingular and if $|\lambda| < \|A^{-1}\|^{-1}$ then λ is not an eigenvalue of A. Here the norm can be any subordinate matrix norm.

In problem 4.5 #19 I showed that if A is invertible then $\|Ax\| \ge \|x\|\|A^{-1}\|^{-1}$.

Suppose that λ were an eigenvalue of A. Then,

$\|Ax\| = \|\lambda x\| = \lambda\|x\|$
$but \quad because \quad \|Ax\| \ge \|x\|\|A^{-1}\|^{-1}$
$\lambda\|x\| \ge \|x\|\|A^{-1}\|^{-1}$
$\lambda \ge \|A^{-1}\|^{-1}$

This contradicts the initial assumption that $|\lambda| < \|A^{-1}\|^{-1}$, so then λ is must not be an eigenvalue of A. QED.