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Numerical Analysis Homework 2a

Kincaid 3.4 #3) Prove that if F is a continuous map of $[a,b]$ into $[a,b]$, then F must have a fixed point. Then determine whether this assertion is true for functions from R to R.

Suppose that F neither starts at a or ends at b (this would mean that F trivially has a fixed point). Then, at some point $F(x \in (a,b]) = a$. However, for this to be the case, the continuous function $F(x)$ must pass through the line $y = x$ since $F(a) > a$ is above this line but $F(x \in (a,b]) = a$ is below this line. At this crossing point, the function F must have a fixed point.

This assertion is not true for functions from R to R. A specific counterexample is $F(x) = x - 1$.

Kincaid 3.4 #6) Consider an iteration function of the form $F(x) = x + f(x)g(x)$, where $f(r) = 0$ and $f'(r) \neq 0$. Find the precise conditions on the function g so that the method of functional iteration will converge cubically to r if started near r.

In this case, then, $F(x) - x = f(x)g(x)$ implies that since $f(r) = 0$ that $F(r) - r = 0$ and r is a fixed point of F.

In the method of functional iteration, $x_{n+1} = F(x_n)$.

Defining $e_n \equiv x_n - r$, then,
$e_{n+1} = x_{n+1} - r = F(x_n) - F(r)$.

Expanding in Taylor series, and taking the limit assuming this method starts sufficiently close to converge:

$$= \left[F(r) + e_n F'(r) + \frac{1}{2} e_n^2 F''(r) + \frac{1}{6} e_n^3 F'''(\zeta) \right] - F(r)$$

$$\to e_n F'(r) + \frac{1}{2} e_n^2 F''(r) + \frac{1}{6} e_n^3 F'''(r)$$

Now I want $F'(r) = F''(r) = 0$ in order to ensure cubic convergence.

Then:

$$F'(r)=0 \to \frac{\partial}{\partial r}\left[r+f(r)g(r)\right]=0 \to f'(r)g(r)=-1$$
$$F''(r)=0 \to \frac{\partial^2}{\partial r^2}\left[r+f(r)g(r)\right]=0 \to 2f'(r)g'(r)+f''(r)g(r)=0$$
$$F'''(r)\neq 0 \to \frac{\partial^3}{\partial r^3}\left[r+f(r)g(r)\right]\neq 0 \to f'''(r)g(r)+3f''(r)g'(r)+3f'(r)g''(r)\neq 0$$

in order for cubic convergence to occur.

Kincaid 3.4 #9) What special properties must a function f have if Newton's method applied to f converges cubically to a zero of f ?

Newton's method has $F(x)=x-\frac{f(x)}{f'(x)}$. I recycle the result from problem #6 above, and call $g(x)=-\frac{1}{f'(x)}$. Now I have:

$$F'(r)=0 \to \frac{\partial}{\partial r}\left[r+f(r)g(r)\right]=0 \to -f'(r)\frac{1}{f'(r)}=-1$$
$$(automatically \quad satisfied)$$
$$F''(r)=0 \to \frac{\partial^2}{\partial r^2}\left[r+f(r)g(r)\right]=0 \to 2f'(r)\frac{f''(r)}{f'(r)^2}-\frac{f''(r)}{f'(r)}=0 \to \frac{f''(r)}{f'(r)}=0 \to f''(r)=0$$
$$F'''(r)\neq 0 \to \frac{\partial^3}{\partial r^3}\left[r+f(r)g(r)\right]\neq 0 \to -\frac{f'''(r)}{f'(r)}+3\frac{f''(r)^2}{f'(r)^2}+3f'(r)\left[\frac{f'''(r)}{f'(r)^2}-2\frac{f''(r)^2}{f'(r)^3}\right]\neq 0$$
$$\to 2\frac{f'''(r)}{f'(r)}+\frac{f''(r)^2}{f'(r)^2}\neq 0 \to f'''(r)\neq 0$$

So I see that by satisfying only $f''(r)=0$ and $f'(r)\neq 0$, a function is guaranteed to converge at cubically or faster using Newton's method. In order to converge exactly cubically, the function also needs $f'''(r)\neq 0$

Kincaid 3.4 #12) Let p be a positive number. What is the value of the following expression? $x=\sqrt{p+\sqrt{p+\sqrt{p+}+\ldots}}$

Note that this can be interpreted as meaning $x=\lim_{n\to\infty}x_n$ where $x_1=\sqrt{p}$, $x_2=\sqrt{p+\sqrt{p}}$ and so forth.

$$x = \sqrt{p + \sqrt{p + \sqrt{p +} + \ldots}}$$

$$x = \sqrt{p + x}$$

$$x^2 = p + x$$

$$x^2 - x - p = 0$$

$$x = \frac{1 \pm \sqrt{1 + 4p}}{2}$$

I select the positive root: $x = \dfrac{1 + \sqrt{1 + 4p}}{2}$

Kincaid 3.4 #13) Let $p > 1$. What is the value of the following continued fraction?

$$x = \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \ldots}}}$$

Use the ideas of the preceding problem to solve this one. Prove that the sequence of values converges by using the Contractive Mapping Theorem.

$$x = \frac{1}{p + x}$$

$$x(p + x) = 1$$

$$x^2 + px - 1 = 0$$

$$x = \frac{-p \pm \sqrt{p^2 + 4}}{2}$$

Since p is positive, I select the positive root:

$$x = \frac{-p + \sqrt{p^2 + 4}}{2}$$

Using the Contractive Mapping Theorem, it is clear that:

$x_{n+1} = \dfrac{1}{p + x_n}$ with $p > 1$ maps every positive number into the space $[0,1]$, and so maps $[0,1]$ into $[0,1]$. Or,

$$F(x) = \frac{1}{p + x}$$

$$|F(x) - F(y)| = \left| \frac{1}{p+x} - \frac{1}{p+y} \right| = \left| \frac{x-y}{(p+x)(p+y)} \right| \le \frac{1}{p^2}|x - y|$$

$$For \quad x, y \in [0, \infty)$$

$$Note: \frac{1}{p^2} < 1 \quad for \quad p > 1$$

So that $F(x)$ is a contractive map by definition.

The contractive mapping theorem states that all contractive maps of a closed domain into the same domain has a unique fixed point which is the limit of every sequence obtained from equation obtained from $x_{n+1} = F(x_n)$ with x_0 in the domain.

Taking $x_0 = p$, then, by the analogy that $x_{n+1} = F(x_n)$ and $F(x) = \dfrac{1}{p+x}$ is a contractive map, the first step will map x_1 into $[0,1]$ and each subsequent step maps into $[0,1]$ so that using x_1 as the starting point, the sequence obtained from this procedure has the limit of the fixed point of $F(x)$, $x = \dfrac{-p + \sqrt{p^2 + 4}}{2}$.

Kincaid 3.4 #40) Show that the following method has third-order convergence for computing $\sqrt{R}$: $x_{n+1} = \dfrac{x_n(x_n^2 + 3R)}{3x_n^2 + R}$.

If I write $f(x) = x^2 - R$, then $\sqrt{R}$ is a zero of this function. In this case, then, I expect that $\sqrt{R}$ is a fixed point of the function

$$F(r) = \frac{r(r^2 + 3R)}{3r^2 + R}$$

$$i.e., F(\sqrt{R}) = \frac{\sqrt{R}\left(\sqrt{R}^2 + 3R\right)}{3\sqrt{R}^2 + R} = \frac{\sqrt{R}(R + 3R)}{3R + R} = \sqrt{R}$$

Now expanding about $\sqrt{R}$,

$$e_{n+1} = F(r+e_n) - F(r) = F(r+e_n) - \sqrt{R}$$

$$F(r+e_n) = F(r) + e_n F'(r) + \frac{1}{2} e_n^2 F''(r) + \frac{1}{6} e_n^3 F'''(r) + \ldots$$

$$F(r) = \sqrt{R}$$

$$F'(r) = \frac{\left(r^2+3R\right)}{3r^2+R} + \frac{2r^2}{3r^2+R} - \frac{6r^2\left(r^2+3R\right)}{\left(3r^2+R\right)^2} \to \frac{(R+3R)}{3R+R} + \frac{2R}{3R+R} - \frac{6R(R+3R)}{(3R+R)^2} = 1 + \frac{1}{2} - \frac{6}{4} = 0$$

$$F''(r) = \frac{2r}{3r^2+R} - \frac{6r\left(r^2+3R\right)}{\left(3r^2+R\right)^2} + \frac{4r}{3r^2+R} - \frac{12r^3}{\left(3r^2+R\right)^2} - \frac{12r\left(r^2+3R\right)}{\left(3r^2+R\right)^2} - \frac{12r^3}{\left(3r^2+R\right)^2} + 2\frac{36r^3\left(r^2+3R\right)}{\left(3r^2+R\right)^3}$$

$$\to \frac{2\sqrt{R}}{3R+R} - \frac{6\sqrt{R}(R+3R)}{(3R+R)^2} + \frac{4\sqrt{R}}{3R+R} - \frac{12R\sqrt{R}}{(3R+R)^2} - \frac{12\sqrt{R}(R+3R)}{(3R+R)^2} - \frac{12R\sqrt{R}}{(3R+R)^2} + 2\frac{36R\sqrt{R}(R+3R)}{(3R+R)^3}$$

$$= \frac{2\sqrt{R}}{4R} - \frac{6\sqrt{R}}{4R} + \frac{4\sqrt{R}}{4R} - \frac{3\sqrt{R}}{4R} - \frac{12\sqrt{R}}{4R} - \frac{3\sqrt{R}}{4R} + \frac{18R\sqrt{R}}{4R}$$

$$= \frac{\sqrt{R}}{4R}(2 - 6 + 4 - 3 - 12 - 3 + 18) = 0$$

So now I see based on this expansion that the highest possible order of convergence is cubic, since the first and second derivatives if the expansion in the error terms vanish at $\sqrt{R}$.