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Numerical Analysis Homework 1

Kincaid 1.1 #29) Develop the Taylor series for $f(x)=\ln x$ about e, writing the results in summation notation and giving the remainder term. Suppose $|x-e|<1$ and accuracy 0.05 is desired. What is the minimum number of terms in the series required to achieve this accuracy?

$$f(x)=1+\frac{1}{e}(x-e)-\frac{1}{2}\frac{1}{e^2}(x-e)^2+\frac{1}{6}\frac{2}{e^3}(x-e)^3+\ldots$$

$$=1+\sum_{i=1}^{n}\frac{(-1)^{i-1}}{ie^i}(x-e)^i+E_n(x)$$

$$E_n(x)=\frac{(-1)^n}{n\xi^{n+1}}(x-e)^{n+1}$$

Then,

$$\max|E_n(x)|=\left|\frac{(-1)^n}{n\xi^{n+1}}(x-e)^{n+1}\right|=\left|\frac{1}{n(e-1)^{n+1}}\right|$$

$$\left|\frac{1}{n(e-1)^{n+1}}\right|<0.05$$

$$when \quad n=3$$

Kincaid 1.1 #32) First develop the function $\sqrt{x}$ in a series of powers of $(x-1)$ and use it to approximate $\sqrt{0.9999999995}$ to ten decimal places.

$$f(x)=1+\frac{1}{2}(x-1)-\frac{1}{2}\cdot\frac{3}{4}(x-1)^2+\frac{1}{6}\cdot\frac{15}{8}(x-1)^3+\ldots$$

$$=1+\sum_{i=1}^{n}\frac{1}{i!}\frac{(2i-1)!!}{2^i}(x-1)^i+E_n(x)$$

$$E_n(x)=\frac{1}{(n+1)!}\frac{(2n+1)!!}{2^{n+1}\xi^{n+\frac{1}{2}}}(x-1)^{n+1}$$

The error at this position is then:

$$f(x)=1+\frac{1}{2}(x-1)-\frac{1}{2}\cdot\frac{1}{4}(x-1)^2+\frac{1}{6}\cdot\frac{3}{8}(x-1)^3+\ldots$$
$$=1+\frac{1}{2}(x-1)+\sum_{i=2}^{n}\frac{(-1)^n}{i!}\frac{(2(i-1)-1)!!}{2^i}(x-1)^i+E_n(x)$$
$$(n\geq 2):E_n(x)=\frac{1}{(n+1)!}\frac{(2n-1)!!}{2^{n+1}(0.9999999995)^{n+\frac{1}{2}}}(0.0000000005)^{n+1}$$

Clearly, $n=0$ is accurate to at least nine decimal places. $n=1$, then, gives the desired accuracy:

$$\sqrt{0.9999999995}\approx 1-\frac{1}{2}(0.0000000005)=0.9999999998$$

Kincaid 1.2 #2) Let a sequence x_n be defined inductively by $x_{n+1}=F(x_n)$. Suppose that $x_n\to x$ as $n\to\infty$ and $F'(x)=0$. Show that $x_{n+2}-x_{n+1}=o(x_{n+1}-x_n)$.

Assume that F is a continuously differentiable function.

Intuitively from the given that this function converges, we would certainly expect that after some n, $x_{n+2}-x_{n+1}<\varepsilon_n(x_{n+1}-x_n)$ where $\varepsilon_n\to 0$ so that $x_{n+2}-x_{n+1}=o(x_{n+1}-x_n)$.

For proof, take the mean value theorem $f(y)=f(c)+f'(\xi)(y-c)$ where ξ is between y and c. Now,

$$f(y)-f(c)=f'(\xi)(y-c)$$
$$\mathit{Identify}\quad c=x_n\quad y=x_{n+1}=F(x_n)=F(c)\quad f=F$$
$$F(x_{n+1})-F(x_n)=F'(\xi)(x_{n+1}-x_n)$$
$$x_{n+2}-x_{n+1}=F'(\xi)(x_{n+1}-x_n)$$

However, since $x_n\to x$ and $F'(x)=0$, it must be the case that $F'(\xi)=\varepsilon_n\to 0$ for a point between $x_n=c$ and $x_{n+1}=y$ with the same slope as the line connecting (x_n,x_{n+1}) and (x_{n+1},x_{n+2}). Now I see that $F'(\xi)=\varepsilon_n\to 0$, satisfying the little-O criterion and so

$x_{n+2}-x_{n+1}=o(x_{n+1}-x_n)$. QED.

Kincaid 3.1 #14) Let the bisection method be applied to a continuous function, resulting in intervals $[a_0,b_0]$, and so on. Let $r=\lim_{n\to\infty}a_n$. Which of these statements can be false?

a) $a_0\leq a_1\leq a_2\leq\ldots$

This statement is certainly true. As the bisection method narrows the interval, no entry a_n may ever be smaller than the previous.

b) $\left|r - 2^{-1}(a_n + b_n)\right| \le 2^{-n}(b_0 - a_0) \quad (n \ge 0)$

This claims that the distance between the midpoint of the interval and the actual root is less than $2^{-n}(b_0 - a_0)$, the size of the interval at this step. This must always be true since the root must lie in the interval, and certainly could never be more than the size of the interval away from the midpoint. In fact, this would still be true if the RHS read $2^{-n+1}(b_0 - a_0)$.

c) $\left|r - 2^{-1}(a_{n+1} + b_{n+1})\right| \le \left|r - 2^{-1}(a_n + b_n)\right| \quad (n \ge 0)$

This claims that the estimate of the root on a subsequent step must be more accurate than the estimate on the previous step. This is not necessarily true; for example, consider a root very near but below the midpoint of the interval on one step. On the next step, the midpoint of the new interval will be approximately half of the new interval's size away from the actual root.

d) $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \quad (n \ge 0)$

Certainly, each interval on subsequent steps can only be a subset of the previous interval since half of the previous interval is cropped on each step.

e) $\left|r - a_n\right| = O(2^{-n}) \quad as \quad n \to \infty$

Certainly, since each step divides by two the interval within which the root may lie, it must be the case that $\left|r - a_n\right| = O(2^{-n}) \quad as \quad n \to \infty$. Specifically, the constant is:

$\left|r - a_n\right| \le (b_0 - a_0)2^{-n} \quad as \quad n \to \infty$

f) $\left|r - c_n\right| < \left|r - c_{n-1}\right| \quad where \quad (n \ge 1)$

This is a similar question to (c), lacking only the "equal" possibility. It is false for the same reason.

Kincaid 3.1 #16) Suppose that $\left|a_n - b_n\right| \le \lambda_n \left|a_{n-1} - b_{n-1}\right|$ for all n with $\lambda_n < 1$. Find an upper bound on $\left|a_n - b_n\right|$ in terms of $\left|a_0 - b_0\right|$ and $\lambda = \max_{1 \le i \le n}\{\lambda_i\}$.

Simply put, then,

$\left|a_n - b_n\right| \le \lambda_n \left|a_{n-1} - b_{n-1}\right| \le \lambda \left|a_{n-1} - b_{n-1}\right|$

$\left|a_n - b_n\right| \le \lambda^n \left|a_0 - b_0\right|$

Kincaid 3.2 #14) Suppose that r **is a double zero of the function** f**. Thus,** $f(r) = f'(r) = 0 \neq f''(r)$**. Show that if** f'' **is continuous, then in Newton's method we shall have** $e_{n+1} \approx \frac{1}{2} e_n$ **(linear convergence).**

$$take: \; f'(x_n) \neq 0$$
$$e_n \equiv x_n - r$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$then:$$
$$e_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - r = \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)}$$
$$Taylor: 0 = f(r) = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\xi_n) \quad \xi_n \in [r, x_n]$$
$$and: 0 = f'(r) = f'(x_n - e_n) = f'(x_n) - e_n f''(\xi'_n) \quad \xi_n \in [r, x_n]$$
$$then:$$
$$\frac{1}{2} e_n^2 f''(\xi_n) = e_n f'(x_n) - f(x_n)$$
$$e_n f''(\xi'_n) = f'(x_n)$$
$$so:$$
$$e_{n+1} = \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)} = \frac{\frac{1}{2} e_n^2 f''(\xi_n)}{f'(x_n)} = \frac{\frac{1}{2} e_n^2 f''(\xi_n)}{e_n f''(\xi'_n)} = \frac{1}{2} \frac{f''(\xi_n)}{f''(\xi'_n)} e_n \approx \frac{1}{2} e_n$$

Kincaid 3.2 #15) Consider a variation of Newton's method in which only one derivative is needed; that is, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}$**. Find** C **and** s **such that** $e_{n+1} = C e_n^s$**.**

$$e_n \equiv x_n - r$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}$$

$$then:$$

$$e_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)} - r = \frac{e_n f'(x_0) - f(x_n)}{f'(x_0)}$$

$$Taylor: 0 = f(r) = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\xi_n) \quad \xi_n \in [r, x_n]$$

$$e.g. \quad e_n f'(x_n) - f(x_n) = \frac{1}{2} e_n^2 f''(\xi_n)$$

$$and: 0 = f(r) = f(x_n - e_n) = f(x_n) - e_n f'(\xi_n) \quad \xi_n \in [r, x_n]$$

$$e.g. \quad f(x_n) = e_n f'(\xi_n)$$

$$e_1 = \frac{e_n f'(x_0) - f(x_0)}{f'(x_0)} = \frac{\frac{1}{2} e_0^2 f''(\xi_0)}{f'(x_0)} \approx \frac{1}{2} e_0^2 \frac{f''(r)}{f'(r)}$$

$$others:$$

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(x_0)} = e_n - \frac{e_n f'(\xi_n)}{f'(x_0)} \approx e_n \left(1 - \frac{f'(r)}{f'(x_0)}\right)$$

$$Thus:$$

$$s = 1$$

$$C = \left(1 - \frac{f'(r)}{f'(x_0)}\right)$$

Kincaid 3.2 #16) Prove that Newton's iteration will diverge for these functions, no matter what (real) starting point is selected.

a) $f(x) = x^2 + 1$

Notably, I see that this function has no zero.

Suppose that this function converges to some r. If this is the case, then by definition of convergence some neighborhood must exist within which $|x_{n+1} - x_n| < \varepsilon$ for all $\varepsilon > 0$ for some n.

Now consider some starting point x_0.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 + 1}{2x_0} = x_0 - \frac{x_0^2 + 1}{2x_0} = \frac{1}{2}x_0 - \frac{1}{2x_0}$$
$$|x_1 - x_0| = \left|\frac{1}{2}x_0 - \frac{1}{2x_0} - x_0\right| = \left|\frac{1}{2x_0} + x_0\right|$$

Note, however, that
$\left|\frac{1}{2x_0} + x_0\right| \geq \sqrt{\frac{1}{2}}$ for all $x_0 \in R$.

Thus, since $|x_{n+1} - x_n| \geq \sqrt{\frac{1}{2}}$, it cannot be the case that $|x_{n+1} - x_n| < \varepsilon$ for all $\varepsilon > 0$ for any choice of starting point, and so Newton's method will never converge.

b) $f(x) = 7x^4 + 3x^2 + \pi$

Again, I see that this function has no zero.

Suppose that this function converges to some r. If this is the case, then by definition of convergence some neighborhood must exist within which $|x_{n+1} - x_n| < \varepsilon$ for all $\varepsilon > 0$ for some n.

Now consider some starting point x_0.
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{7x_0^4 + 3x_0^2 + \pi}{28x_0^3 + 6x_0} = x_0 - \frac{x_0^2 + 1}{2x_0} = \frac{1}{2}x_0 - \frac{1}{2x_0}$$
$$|x_1 - x_0| = \left|-\frac{7x_0^4 + 3x_0^2 + \pi}{28x_0^3 + 6x_0}\right| = \left|\frac{7x_0^4 + 3x_0^2 + \pi}{28x_0^3 + 6x_0}\right|$$

This is zero whenever the numerator, $7x_0^4 + 3x_0^2 + \pi$, is zero. However, this function is nowhere zero and therefore

$\left|\frac{7x_0^4 + 3x_0^2 + \pi}{28x_0^3 + 6x_0}\right| \geq \varepsilon$ for some $\varepsilon > 0$ for all x_0 and therefore by $|x_{n+1} - x_n| > \varepsilon > 0$ for all x_0, so that convergence cannot be obtained by Newton's method.

Kincaid 3.2 #19) Prove that if r is a zero of multiplicity k of the function f, then quadratic convergence in Newton's iteration will b e restored by making this modification: $x_{n+1} = x_n - k\frac{f(x_n)}{f'(x_n)}$.

$$e_n \equiv x_n - r$$

$$x_{n+1} = x_n - k\frac{f(x_n)}{f'(x_n)}$$

then:

$$e_{n+1} = x_n - k\frac{f(x_n)}{f'(x_n)} - r = e_n - k\frac{f(x_n)}{f'(x_n)}$$

$$Taylor: 0 = f(r) = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \ldots + \frac{(-1)^k}{k!} e_n^k f^{(k)}(x_n) + \frac{(-1)^{k+1}}{(k+1)!} e_n^{k+1} f^{(k+1)}(\xi_n) \quad \xi_n \in [r, x_n]$$

$$e.g. \quad -f(x_n) = \frac{(-1)^k}{k!} e_n^k f^{(k)}(\xi_n) + \frac{(-1)^{k+1}}{(k+1)!} e_n^{k+1} f^{(k+1)}(\xi_n)$$

$$Taylor: 0 = f'(r) = f'(x_n - e_n) = f'(x_n) - e_n f''(x_n) + \ldots + \frac{(-1)^{k-1}}{(k-1)!} e_n^{k-1} f^{(k)}(\xi_n) \quad \xi_n \in [r, x_n]$$

$$e.g. \quad -f'(x_n) = -\frac{(-1)^{k-1}}{(k-1)!} e_n^{k-1} f^{(k)}(\xi_n)$$

Combining these, I get:

$$e_{n+1} = e_n + k\frac{\frac{(-1)^k}{k!} e_n^k f^{(k)}(x_n) + \frac{(-1)^{k+1}}{(k+1)!} e_n^{k+1} f^{(k+1)}(\xi_n)}{\frac{(-1)^{k-1}}{(k-1)!} e_n^{k-1} f^{(k)}(\xi_n)}$$

$$take \quad x_n \to r \quad \xi_n \to r$$

$$= e_n - e_n + \frac{e_n^2}{k+1}\frac{f^{(k+1)}(\xi_n)}{f^{(k)}(\xi_n)} = \frac{e_n^2}{k+1}\frac{f^{(k+1)}(\xi_n)}{f^{(k)}(\xi_n)} = Ce_n^2$$

just as expected.

Kincaid 3.2 #21) Halley's Method for solving the equation $f(x) = 0$ uses the iteration formula $x_{n+1} = x_n - \frac{f_n f_n'}{(f_n')^2 - \frac{1}{2}(f_n f_n'')}$, where $f_n = f(x_n)$ and so on. Show that this formula results when Newton's iteration is applied to the function $\frac{f}{\sqrt{f'}}$.

In Newton's method, $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$. Then, if $F(x_n) = \frac{f_n}{\sqrt{f_n'}}$:

$$F'(x_n) = \frac{f'_n}{\sqrt{f'_n}} - \frac{1}{2}\frac{f_n f_n''}{(f_n')^{\frac{3}{2}}}$$

$$\frac{F(x_n)}{F'(x_n)} = \frac{\dfrac{f_n}{\sqrt{f'_n}}}{\dfrac{f'_n}{\sqrt{f'_n}} - \dfrac{1}{2}\dfrac{f_n f_n''}{(f_n')^{\frac{3}{2}}}} = \frac{f_n}{f'_n - \dfrac{1}{2}\dfrac{f_n f_n''}{f_n'}} = \frac{f_n f_n'}{(f'_n)^2 - \dfrac{1}{2} f_n f_n''}$$