Magnetic Monopole Problems

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Topic Outline:

- Since 1931 when Dirac showed that monopoles implied charge quantization, there has been recurring effort to either show the existence of magnetic monopoles or show that they do not exist.
- What are magnetic monopoles?
    - Dirac String formalism.
        - Necessary to avoid magnetic scalar potential/vector potential.
        - What is current distribution that produces this?

            Strictly, it is seen as a chain of magnetic dipoles.

            If $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$,

            Then we may expand the expression as:

            $$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

            Or, expanding this into monopole, dipole, etc. moments,

            $$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\vec{m}(\vec{x}')\times\vec{x}'}{|\vec{x}-\vec{x}'|^3} + ... d^3x'$$

            (this is the only form that can work.)

            Taking only the magnetic dipole part:

            $$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{m}(\vec{x}')\times\vec{x}'}{|\vec{x}-\vec{x}'|^3} d^3x' \qquad \textit{Important.}$$

            And clearly $\vec{m}(\vec{x}') \propto g$ along the Dirac string.

        - What are Maxwell's Laws in the presence of monopoles?

            $$\vec{\nabla}\cdot\vec{D} = \rho_e \qquad \vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J}_e$$

            $$\vec{\nabla}\cdot\vec{B} = \rho_m \qquad -\vec{\nabla}\times\vec{E} = \frac{\partial\vec{B}}{\partial t} + \vec{J}_m \qquad \textit{Important}$$

            $$\vec{D} = \varepsilon_0\vec{D} \qquad \vec{H} = \frac{1}{\mu_0}\vec{D}$$

            (See Jackson 6.17 for more)

        - How do you reconcile the ambiguity in terms of the Dirac string?

            Gauge problems:
            (Jackson 6.19)
            (Teplitz 8.3)
- What are their properties?
    - Charge

        - -Jackson derivation *Easiest Math.*
            - (Jackson 6.17)
        - -Angular Momentum in Fields *Clearest.*
            - (Teplitz 8.2)
        - -Wu-Yang derivation connecting gauges
            - (Teplitz 8.7)
        - -Aharonov-Bohm effect
            - (Detectablility of vector potential)
- -Lifetime
    - -Indivisibility of magnetic charge—should be infinite
- -Mass

    Skeptical that it might be bigger than the Planck mass: $\sqrt{\frac{\hbar c}{G}} = 1.22 \cdot 10^{19} \frac{GeV}{c^2}$, the mass at which the particle would be eternally collapsing on itself.

    (**Relevant plot**: Search for a Flux of cosmic-ray magnetic monopoles page 44)

    Given galactic halo mass limit, number should limited by missing mass, maximum flux, average velocity.
- -How do they interact with matter?
    - -Binding
        - -Nuclear
            - (Teplitz 8.16)
            - Why wouldn't it participate in this kind of binding?
            - -Centrifugal Barrier
            - -Paramagnetism like $\frac{1}{r^4}$ (induced moment like B again, field like inverse-square r)
            - -Diamagnetic Interaction at extreme close distance like $\frac{1}{r^6}$.
        - -Molecular
            - (Teplitz 8.18)
            - -Here, Bohr magneton is largely fixed and paramagnetic is like $\frac{1}{r^2}$, diamagnetic like $\frac{1}{r^4}$ (square in B field).
        - -Ferromagnets
            - (Teplitz 8.19)
            - -Force like $\frac{g^2}{r^2}$, potential like $\frac{g^2}{r}$ at atomic distances.
    - -Energy/Ionization Losses

-Lorentz Boost Equations *Important.*

$$F^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}$$

$$\Lambda^{\mu}{}_{\alpha} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

-Ionization should be higher at high velocity, by the Lorentz boost on the magnetic fields, opposite that of a electron.

(Teplitz 8.12)

Constant:

$$\left(\frac{dE}{dx}\right)_q = \frac{4\pi NZg^2e^2}{m_ec^2}\left\{\ln\left[\frac{1.123\gamma^2m_ec^3\beta^2}{ge\langle\omega\rangle}\right]\right\}$$

Linear in NZ since the density of electrons is linear in this.
g and e should both be squared, since it should not matter whether these are positive or negative.
As mass of electron increases, energies of harmonics decrease ($\omega = \sqrt{\frac{k}{m}}$)
Should increase in β as electric fields increase.

(Relevant Plot: End of section two in Search for Magnetic Monopoles, Q-Balls…)

-How many are there?

(See **Relevant Plot** above)

-Parker Bound

-Post-creation annihilation is negligible. These particles have relatively high velocity, low density, and charge only an order of magnitude higher than electron.
-At fast speeds and mass like Planck mass, or any case where magnetic defelction is difficult the best case is $10^{-13}cm^{-2}sr^{-1}s^{-1}$.
-At slow speeds or lower mass, magnetic deflection is easy and the best case is closer to $10^{-15}cm^{-2}sr^{-1}s^{-1}$.
(See Magnetic Monopoles and Survival… for calculation details.)
-Simplest argument (survival of galactic magnetic fields)

Flux is less than $10^{-16} cm^{-2} sr^{-1} s^{-1}$

$\tau_{regen} = \frac{\frac{B^2}{8\pi}}{\vec{j}_m \cdot \vec{B}}$, with B on the order of microGauss and regeneration on the order of 10 million years.

-Missing Mass Bound

Like: $\frac{MassDeficit}{MassMonopole} \cdot \frac{VelocityMonopole}{4\pi VolumeGalaxy}$. For the our galaxy, at about 1/1000 the speed of light, at the Planck mass, about $10^{-13} cm^{-2} sr^{-1} s^{-1}$

-Where do we look for them / how? (3 experiments)

- -Could subatomic particles be made of them?
  - -No, by the hyperfine splitting of a proton (Teplitz 8.9, 8,10, 8.11, p 317)
  - *Important: Draw Field*
- -In Accelerators
  - -Purcell Experiment/Paper: Capture in oil with rapid braking, use slight field to pump up to scintillator.
  - -Cross-section for production is placed at the order of $10^{-34} - 10^{-40} cm^2$ for production by 30 BeV protons with nuclei (maximum mass about 2.9 BeV). Essentially, this does not happen.
  - -In our picture, nuclear binding was not an issue. The aggregate object would be stripped and its ionization dominated by the monopole.
  - -Binding with nucleus at a range intermediate between nuclear and atomic, or if it was not of unit charge.
  - -Only issues
  - In other words, this is not likely to happen.
- -In Moon Rocks
  - -Superconducting Ring Detector
  - -(Ampere's Law Example—Show independent of velocity of particle travel)
  - -Importance of Heavy Nuclei as a momentum dump
  - -Upper Flux Limit from Cosmic Monopoles: $10^{-18} cm^{-2} s^{-1} sr^{-1}$
  - -Upper Flux Limit on Cosmic Ray Production via collision with nucleons: Very poor for high Monopole masses (enormous)
  - -Results questionable
    - -Assume good trapping in moon rocks near surface
    - -Assume poor migration
    - -Typical energy loss = 10 GeV cm^2/g
    - -Essentially depends on nuclear trapping.

- Cosmic Ray or Galactic Flux
  - Carberra paper. Similar to Moon Rocks, really. Flux limits probably most important. On the order of $7.3 \cdot 10^{-13} cm^{-2} sr^{-1} s^{-1}$ for any source.
- Why would we like to see magnetic monopoles?
  - Consequences:
    - Time Inversion Symmetry
      - Magnetic Charge must be odd under time reversal, by the example of a charge circulated in a field generated this way.
      - Magnetic Current must therefore be even: an opposite charge traveling the opposite direction.
      - Thus this is no longer a valid symmetry of the laws of physics!
    - Space Inversion Symmetry
      - Imagine proton going around in a loop in a constant B field created by monopoles.
      - Symmetry over an axis in-plane:
        - Proton is now going the opposite direction… but it will not retrace its original path! Charge odd under spatial inversion too.
      - Symmetry over an axis out of plane:
        - The B field has now switched! The proton will switch loops again!
  - Time and space inversion are apparently slightly violated in the weak interaction and this could provide a mechanism.
  - No good explanation as to why not.
  - GUT wants them in some manifestations
  - Possible explanation for dark matter
  - Explain charge quantization.
- In Conclusion
  - The search for magnetic monopoles has yet been unsuccessful, but is likely to continue until a theoretical explanation as to why they should not exist is forthcoming. If found, it would have serious ramifications on our understanding of Physics.

(Jackson 6.16)

a) **Calculate the force in newtons acting on a Dirac monopole of the minimum magnetic charge located a distance 0.5 angstroms and in the median plane of a magnetic dipole with dipole moment equal to one nuclear magneton $e\hbar / 2m_p$.**

$$F = gH = g\mu_0 \frac{\mu}{r^3}$$

Use the minimum value of $g = \frac{2\pi\hbar}{e}$ so that $F = \mu_0 \frac{2\pi\hbar}{e}\frac{e\hbar}{2m_p}\frac{1}{r^3} = \mu_0 \frac{\pi\hbar^2}{m_p}\frac{1}{r^3}$.

Then, the magnetic field at this position should be

$$H = \left(4\pi \cdot 10^{-7}\,\frac{N}{A^2}\right)\frac{(1.60\cdot 10^{-19}\,C)(1.05\cdot 10^{-34}\,Js)}{2\cdot 1.67\cdot 10^{-27}\,kg}\frac{1}{(0.5\cdot 10^{-10}\,m)^3}$$

$$H = 0.051 \quad T$$

Now I expect that $F = gH$ is in Newtons, so that g has units $\frac{N}{T}$.

For a Dirac monopole, I have $g = \frac{2\pi\hbar}{e}$, which is in units $\frac{Js}{C}$.

Converting this to SI, I need some additional factors:

$$\frac{\left(\frac{N}{T}\right)}{\left(\frac{Js}{C}\right)} = \frac{NC}{TJs} = \frac{CAm}{Js} = \frac{Am}{Vs} = \frac{c}{Z} = \frac{c}{\mu_0 c} = \frac{1}{\mu_0}$$

Then, in SI units

$$g = \frac{2\pi\hbar}{e\mu_0} = \frac{2\pi(1.05\cdot 10^{-34}\,Js)}{(1.60\cdot 10^{-19}\,C)\left(4\pi\cdot 10^{-7}\,\frac{N}{A^2}\right)} = 3.28\cdot 10^{-9} \quad \frac{N}{T}$$

and so

$$F = gH = 0.051\cdot 3.28\cdot 10^{-9}\,N = 1.70\cdot 10^{-10}\,N.$$

b) **Compare the force in part (a) with atomic forces such as the direct electrostatic force between charges (at the same separation), the spin-orbit force, and the hyperfine interaction. Comment on the question of binding of magnetic monopoles to nuclei with magnetic moments. Assume that the monopole mass is at least that of a proton.**

The electrostatic force between two charges at this separation is

$$F = qE = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{(0.5\cdot 10^{-10}\,m)^2} = 9.216\cdot 10^{-8}\,N.$$

The spin-orbit force between the electron and its own magnetic moment charges at this separation is calculated using:

The Bohr magneton:

$$e\hbar / 2m_e = 9.22\cdot 10^{-24}\,\frac{J}{T}$$

The electron spin magneton:

$$\frac{e}{2m_e} g\left(\frac{1}{2}\hbar\right) = 9.22 \cdot 10^{-24} \frac{J}{T}$$

Orienting these parallel to one another and perpendicular to the orbit, I have:

$$F = \frac{3}{4\pi}\frac{\mu_0}{r^4}(\mu_B \mu_s) = 4.08 \cdot 10^{-12} N$$

Finally, the hyperfine interaction between an electron and a nuclear magneton is

$$\mu_p = e\hbar / 2m_p = 5.03 \cdot 10^{-27} \frac{J}{T}$$

$$F = \frac{3}{4\pi}\frac{\mu_0}{r^4}(\mu_B \mu_p) = 2.23 \cdot 10^{-15} N$$

So, binding with nuclei at the radius of an electron in terms of magnetic moments is not likely as the force is less than 100 times that of the force between a proton and electron but nonetheless stronger than the spin-orbit and hyperfine interactions. However, note that since the force will increase as distance decreases since the interaction force is $F \propto \frac{1}{r^3}$, so a slight decrease in radius will bring the force to the order of magnitude of the proton-electron force. However, the issue is now that if the mass of the magnetic monopole is at least as large as that of the proton then the force must be at least $\frac{m_p}{m_e}$ times larger than the proton-electron force in order to keep the magnetic monopole in orbit. With this in mind, it seems like the binding would be very strong if this occurred. However, I am concerned that given the sensitivity of such an to angular position and to nuclear spin orientation may make such an orbit extremely unstable and further may present quantum uncertainty issues.

(Jackson 6.17)

a) **For a particle possessing both electric and magnetic charges, show that the generalization of the Lorentz force is**
$\vec{F} = q_e\vec{E} + q_m\vec{H} + q_e\vec{v}\times\vec{B} - q_m\vec{v}\times\vec{D}$.

The Lorentz force is typically given by: $F_{el}{}^{\alpha} = q_e u_\beta F^{\alpha\beta}$, where $F^{\alpha\beta}$ is the field strength tensor and $u_\beta = \gamma(c, v_x, v_y, v_z)$. However, under the modified Maxwell equations I would expect similarly that

$$F_{mag}{}^{e} = q_m g^{ed} u_\alpha \varepsilon_{d\alpha\beta c} F^{\beta c}$$

where $g = diag[1,-1,-1,-1]$

for the magnetic parts.

Writing explicitly, I have:

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{bmatrix} \qquad \varepsilon_{d\alpha\beta c}F^{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ B_y & \frac{E_z}{c} & 0 & -\frac{E_x}{c} \\ B_z & -\frac{E_y}{c} & \frac{E_x}{c} & 0 \end{bmatrix}$$

Then,

$$F_{tot}{}^{\alpha} = F_{el}{}^{\alpha} + F_{mag}{}^{\alpha} \to q_e\vec{E} + q_m\vec{H} + q_e\vec{v}\times\vec{B} - q_m\vec{v}\times\vec{D}.$$

More simply, however, by the symmetries of Maxwell's Laws in the presence of Monopoles, I expect the force law for magnetic charges to have the same format as that for electric charges, except so that $\vec{F} = q_m\left(\vec{B} - \vec{v}\times\vec{E}\right)$, with the additional minus sign coming from the minus sign in Faraday's law which is opposite the corresponding sign in Ampere's law.

b) Show that this expression for the force is invariant under a duality transformation of both fields and charges.

The duality transforms are:

$$\vec{E} = \vec{E}'\cos\xi + Z_0\vec{H}'\sin\xi \qquad Z_0\vec{D} = Z_0\vec{D}'\cos\xi + \vec{B}'\sin\xi$$
$$Z_0\vec{H} = -\vec{E}'\sin\xi + Z_0\vec{H}'\cos\xi \qquad \vec{B} = -Z_0\vec{D}'\sin\xi + \vec{B}'\cos\xi$$
$$Z_0\rho_e = Z_0\rho_e'\cos\xi + \rho_m'\sin\xi \qquad Z_0\vec{J}_e = Z_0\vec{J}_e'\cos\xi + \vec{J}_m'\sin\xi$$
$$\rho_m = -Z_0\rho_e'\sin\xi + \rho_m'\cos\xi \qquad \vec{J}_m = -Z_0\vec{J}_e'\sin\xi + \vec{J}_m'\cos\xi$$

Transforming:

$$\vec{F}=\left(\begin{array}{l}\frac{1}{Z_0}\left(Z_0\rho_e'\cos\xi+\rho_m'\sin\xi\right)\left(\vec{E}'\cos\xi+Z_0\vec{H}'\sin\xi\right)\\+\frac{1}{Z_0}\left(-Z_0\rho_e'\sin\xi+\rho_m'\cos\xi\right)\left(-\vec{E}'\sin\xi+Z_0\vec{H}'\cos\xi\right)\\+\frac{1}{Z_0}\left(Z_0\rho_e'\cos\xi+\rho_m'\sin\xi\right)\vec{v}\times\left(-Z_0\vec{D}'\sin\xi+\vec{B}'\cos\xi\right)\\-\frac{1}{Z_0}\left(-Z_0\rho_e'\sin\xi+\rho_m'\cos\xi\right)\vec{v}\times\left(Z_0\vec{D}'\cos\xi+\vec{B}'\sin\xi\right)\end{array}\right)$$

$$=\left(\begin{array}{l}\rho_e'\vec{E}'\cos^2\xi+\frac{1}{Z_0}\rho_m'\vec{E}'\cos\xi\sin\xi+Z_0\rho_e'\vec{H}'\cos\xi\sin\xi+\rho_m'\vec{H}'\sin^2\xi\\+\rho_e'\vec{E}'\sin^2\xi-\frac{1}{Z_0}\rho_m'\vec{E}'\cos\xi\sin\xi-Z_0\rho_e'\vec{H}'\cos\xi\sin\xi+\rho_m'\vec{H}'\cos^2\xi\\-Z_0\rho_e'\vec{v}\times\vec{D}'\cos\xi\sin\xi+\rho_e'\vec{v}\times\vec{B}'\cos^2\xi+\frac{1}{Z_0}\rho_m'\vec{v}\times\vec{B}'\cos\xi\sin\xi-\rho_m'\vec{v}\times\vec{D}'\sin^2\xi\\+Z_0\rho_e'\vec{v}\times\vec{D}'\cos\xi\sin\xi+\rho_e'\vec{v}\times\vec{B}'\sin^2\xi-\frac{1}{Z_0}\rho_m'\vec{v}\times\vec{B}'\cos\xi\sin\xi-\rho_m'\vec{v}\times\vec{D}'\cos^2\xi\end{array}\right)$$

Adding terms, then, I see that force is indeed invariant under Duality:
$\vec{F}=q_e'\vec{E}'+q_m'\vec{H}'+q_e'\vec{v}\times\vec{B}'-q_m\vec{v}'\times\vec{D}'$

c) Show that the Dirac quantization condition is generalized for two particles possessing electric and magnetic charges e_1, g_1 and e_2, g_2 respectively to

$$\frac{e_1g_2-e_2g_1}{\hbar}=2\pi n$$

and that the relation is invariant under a duality transformation of the charges.

First, consider particle 2 (traveling parallel to the z-axis at a large impact parameter b) passing far away from particle 1 (which sits at the origin). Assume that particle 2 is sufficiently far away that it is completely undeflected.

Using Gaussian units and the Lorentz force above,

$$\vec{F} = q_e\vec{E} + q_m\vec{H} + q_e\vec{v}\times\vec{B} - q_m\vec{v}\times\vec{D}$$

$$F_x = q_{e,2}\frac{q_{e,1}r_x}{4\pi r^3} + q_{m,2}\frac{q_{m,1}r_x}{4\pi r^3}$$

$$F_z = q_{e,2}\frac{q_{e,1}r_z}{4\pi r^3} + q_{m,2}\frac{q_{m,1}r_z}{4\pi r^3}$$

$$F_y = q_{e,2}\frac{q_{m,1}vr_x}{4\pi r^3} - q_{m,2}\frac{q_{e,1}vr_x}{4\pi r^3}$$

Now integrating the forces in the x and y dimensions in the undeflected limit,

$$\Delta p_x = \int_{-\infty}^{\infty} F_x dt = \left(\frac{q_{e,2}q_{e,1}}{4\pi} + \frac{q_{m,2}q_{m,1}}{4\pi}\right) b\int_{-\infty}^{\infty}\frac{dt}{\left(b^2+v^2t^2\right)^{\frac{3}{2}}}dt$$

$$= \left(\frac{q_{e,2}q_{e,1}}{4\pi} + \frac{q_{m,2}q_{m,1}}{4\pi}\right) b\frac{2}{vb^2} = \left(\frac{q_{e,2}q_{e,1}}{4\pi} + \frac{q_{m,2}q_{m,1}}{4\pi}\right)\frac{2}{vb}$$

$$\Delta p_y = \int_{-\infty}^{\infty} F_y dt = \left(\frac{q_{e,2}q_{m,1}}{4\pi} - \frac{q_{m,2}q_{e,1}}{4\pi}\right) vb\int_{-\infty}^{\infty}\frac{dt}{\left(b^2+v^2t^2\right)^{\frac{3}{2}}}dt$$

$$= \left(\frac{q_{e,2}q_{m,1}}{4\pi} - \frac{q_{m,2}q_{e,1}}{4\pi}\right) vb\frac{2}{vb^2} = \frac{2}{b}\left(\frac{q_{e,2}q_{m,1}}{4\pi} - \frac{q_{m,2}q_{e,1}}{4\pi}\right)$$

However, notice that in the undeflected limit, v must be relatively large and so I have:

$$\Delta p_x = \left(\frac{q_{e,2}q_{e,1}}{4\pi} + \frac{q_{m,2}q_{m,1}}{4\pi}\right)\frac{2}{vb} \to 0$$

$$\Delta p_y = \frac{2}{b}\left(\frac{q_{e,2}q_{m,1}}{4\pi} - \frac{q_{m,2}q_{e,1}}{4\pi}\right)$$

$$\Delta L_z = b\Delta p_y = 2\left(\frac{q_{e,2}q_{m,1}}{4\pi} - \frac{q_{m,2}q_{e,1}}{4\pi}\right)$$

Now assuming any change in angular momentum must occur in an integer multiple of $\hbar$, since anything rotating has a quantum wave-function that must be a multiple of the same in order to ensure that the wave function is singly valued then

$$\Delta L_z = 2\left(\frac{q_{e,2}q_{m,1}}{4\pi} - \frac{q_{m,2}q_{e,1}}{4\pi}\right) = n\hbar$$

$$\frac{q_{e,2}q_{m,1} - q_{m,2}q_{e,1}}{\hbar} = 2n\pi$$

Checking invariance under duality,

$$\frac{q_{e,2}q_{m,1}-q_{m,2}q_{e,1}}{\hbar}\rightarrow$$

$$\frac{\left(q'_{e,2}\cos\xi+q'_{m,2}\sin\xi\right)\left(q'_{m,1}\cos\xi-q'_{e,1}\sin\xi\right)-\left(q'_{e,1}\cos\xi+q'_{m,1}\sin\xi\right)\left(q'_{m,2}\cos\xi-q'_{e,2}\sin\xi\right)}{\hbar}$$

$$=\frac{\left(q'_{e,2}\,q'_{m,1}\cos^2\xi-q'_{m,2}\,q'_{e,1}\sin^2\xi\right)-\left(q'_{m,2}\,q'_{e,1}\cos^2\xi-q'_{e,2}\,q'_{m,1}\sin^2\xi\right)}{\hbar}$$

$$=\frac{q'_{e,2}\,q'_{m,1}-q'_{m,2}\,q'_{e,1}}{\hbar}$$

(Jackson 6.18) Consider the Dirac expression

$$\vec{A}(\vec{x})=\frac{g}{4\pi}\int_L\frac{d\vec{l}\,'\times(\vec{x}-\vec{x}\,')}{\left|\vec{x}-\vec{x}\,'\right|^3}$$

for the vector potential of a magnetic monopole and its associated string L. Suppose for definitiveness that the monopole is located at the origin and the string along the negative z-axis.

(a) Calculate $\vec{A}$ explicitly and show that in spherical coordinates it has components

$$A_r=0\quad A_\theta=0\quad A_\phi=\frac{g(1-\cos\theta)}{4\pi r\sin\theta}=\left(\frac{g}{4\pi r}\right)\tan\left(\frac{\theta}{2}\right)$$

$$\vec{A}(\vec{x})=\frac{g}{4\pi}\int_{-\infty}^{0}\frac{\hat{k}dz\times\left(r\sin\theta\cos\phi\hat{i}+r\sin\theta\sin\phi\hat{j}+r\cos\theta\hat{k}-z\hat{k}\right)}{\left((r\cos\theta-z)^2+(r\sin\theta)^2\right)^{\frac{3}{2}}}$$

Under some simplification, this becomes:

$$\vec{A}(\vec{x})=\frac{g}{4\pi}\int_{-\infty}^{0}\frac{rdz\hat{j}\sin\theta\cos\phi-rdz\hat{i}\sin\theta\sin\phi}{\left(z^2-2rz\cos\theta+r^2\right)^{\frac{3}{2}}}=\frac{gr\sin\theta\hat{\phi}}{4\pi}\int_{-\infty}^{0}\frac{dz}{\left(z^2-2rz\cos\theta+r^2\right)^{\frac{3}{2}}}$$

According to Mathematica, the result of this integral is

$$\vec{A}(\vec{x})=\frac{gr\hat{\phi}}{4\pi}\left(\frac{\sin\theta}{r^2(1+\cos\theta)}\right)=\frac{g\hat{\phi}}{4\pi r}\tan\left(\frac{\theta}{2}\right)$$

(b) Verify that $\vec{B}=\vec{\nabla}\times\vec{A}$ is the Coulomb-like field of a point charge, except perhaps at $\theta=\pi$.

$$\vec{B} = \vec{\nabla} \times \vec{A} = \hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left[\frac{g}{4\pi r}\tan\frac{\theta}{2}\right]\right) - \hat{\theta}\frac{\partial}{\partial r}\left(r\left[\frac{g}{4\pi r}\tan\frac{\theta}{2}\right]\right)$$

$$= \hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left[\frac{g}{4\pi r}\tan\frac{\theta}{2}\right]\right)$$

$$= \hat{r}\frac{g}{4\pi r^2}\left(\csc\theta\left(\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\sin\theta + \cos\theta\tan\frac{\theta}{2}\right)\right) = \hat{r}\frac{g}{4\pi r^2}$$

except possibly where the secant and tangent diverge, at $\theta = \pi$. Here, I must include a delta-function contribution from the magnetic field being piped up the Dirac string:

$$= \hat{r}\frac{g}{4\pi r^2} + \hat{z}\frac{g}{4\pi}\delta(x)\delta(y)\theta(-z)$$

(c) With the magnetic field determined in part b, evaluate the total magnetic flux passing through the circular loop of radius $R\sin\theta$ shown in the figure. Consider $\theta < \frac{\pi}{2}, \theta > \frac{\pi}{2}$ separately, but always calculate the upward flux.

I notice that this diverges just like an electric field, so I may take on the top part the proportion of the sphere centered at the origin surrounded by this circle, and on the bottom part negative that (since I want upward flux). For utility, I'll take the integral over the surface of the top theta-degrees of a sphere:

$$\int_0^\theta 2\pi R \cdot R\sin\theta' d\theta' = 2\pi R^2(1-\cos\theta).$$

Now in the upper half, I get using Gauss's Law:
$\frac{g}{4\pi R^2}2\pi R^2(1-\cos\theta) = \frac{g}{2\pi}(1-\cos\theta)$, and where $\theta > \frac{\pi}{2}$ I get
$-\frac{g}{2\pi}(1-\cos(\pi-\theta)) = -\frac{g}{2\pi}(1+\cos\theta)$.

(d) From $\oint \vec{A}\cdot d\vec{l}$ around the loop, determine the total magnetic flux through the loop. Compare with the result from part c. Show that they are equal for $\theta < \frac{\pi}{2}$ but have a constant difference for $\theta > \frac{\pi}{2}$. Interpret this difference.

$\left(\frac{g}{4\pi R}\right)\tan\left(\frac{\theta}{2}\right)\cdot 2\pi R\sin\theta = \frac{g}{2\pi}(1-\cos\theta)$, which differs by a constant $\frac{g}{\pi}$ in the lower plane: this is attributable to the flux in the Dirac string which occupies the negative z axis.

(Jackson 6.19)

(a) Apply space inversion to the monopole vector potential of Problem 6.18 and show that the vector potential becomes:

$$A'_\phi = -g\frac{(1+\cos\theta)}{4\pi r\sin\theta} = -\frac{g}{4\pi r}\cot\frac{\theta}{2}$$

with the other components vanishing. Show explicitly that its curl gives the magnetic field of a magnetic monopole except perhaps at $\theta = 0$.

The difference is now that the integral extends down the positive z-axis from infinity and my x and y coordinates are mirrored, so that I have:

$$\vec{A}(\vec{x}) = \frac{g}{4\pi}\int_\infty^0 \frac{\hat{k}dz \times \left(r\sin\theta\cos\phi\hat{i} + r\sin\theta\sin\phi\hat{j} + r\cos\theta\hat{k} - z\hat{k}\right)}{\left((-r\cos\theta + z)^2 + (-r\sin\theta)^2\right)^{\frac{3}{2}}}$$

(Note that the numerator is unchanged because both the spatial and vectorial portions of the coordinate are inverted. The denominator is changed because this involves only spatial portions which are inverted.)

This will simplify to:

$$\vec{A}(\vec{x}) = \frac{g}{4\pi}\int_\infty^0 \frac{rdz\hat{j}\sin\theta\cos\phi - rdz\hat{i}\sin\theta\sin\phi}{\left(z^2 - 2rz\cos\theta + r^2\right)^{\frac{3}{2}}} = \frac{gr\sin\theta\hat{\phi}}{4\pi}\int_\infty^0 \frac{dz}{\left(z^2 - 2rz\cos\theta + r^2\right)^{\frac{3}{2}}}$$

Simplifying in Mathematica, this is:

$$\vec{A}(\vec{x}) = \frac{g}{4\pi}\int_\infty^0 \frac{rdz\hat{j}\sin\theta\cos\phi - rdz\hat{i}\sin\theta\sin\phi}{\left(z^2 - 2rz\cos\theta + r^2\right)^{\frac{3}{2}}} = \frac{gr\sin\theta\hat{\phi}}{4\pi}\left(\frac{1}{r^2(\cos\theta - 1)}\right) = -\frac{g\hat{\phi}}{4\pi r}\cot\left(\frac{\theta}{2}\right)$$

Verifying that this is indeed a monopole field, I have:

$$\begin{aligned}
\vec{B} = \vec{\nabla}\times\vec{A} &= \hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left[-\frac{g}{4\pi r}\cot\frac{\theta}{2}\right]\right) - \hat{\theta}\frac{\partial}{\partial r}\left(r\left[-\frac{g}{4\pi r}\cot\frac{\theta}{2}\right]\right) \\
&= \hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left[-\frac{g}{4\pi r}\cot\frac{\theta}{2}\right]\right) = \hat{r}\frac{1}{r\sin\theta}\left(\cos\theta\cot\frac{\theta}{2} - \frac{1}{2}\csc^2\frac{\theta}{2}\sin\theta\right) \\
&= -\hat{r}\frac{g}{4\pi r^2\sin\theta}(-\sin\theta) = \hat{r}\frac{g}{4\pi r^2}
\end{aligned}$$

except possibly where the cosecant and cotangent diverge, at $\theta = 0$.

(b) Show that the difference, $\delta\vec{A} = \vec{A}' - \vec{A}$, can be expressed as the gradient of a scalar function, indicating that the original and space-inverted vector potentials differ by a gauge transformation.

$$\delta\vec{A} = \vec{A}' - \vec{A} = -\frac{g}{4\pi r}\left[\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right]\hat{\phi}$$

Certainly, I have already seen that there must exist some scalar function that produces this difference since $\vec{B} = \vec{\nabla} \times \vec{A}$ was the same for both problems (6.18) and (6.19) and the only way to create this is to have some scalar component ψ so that $\vec{\nabla} \times \vec{\nabla}\psi = 0$. Exploring so as to find the specific function in question, I have:

$$\delta\vec{A} = \vec{A}' - \vec{A} = -\frac{g}{4\pi r}\left[\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right]\hat{\phi} = -\frac{g}{2\pi r}\csc\theta\hat{\phi}.$$

Examining the form of the $\vec{\nabla}$ operator in spherical coordinates quickly shows that the function in question must in fact be:

$$\psi = -\frac{g}{2\pi}\phi$$

(c) Interpret the gauge function in terms of figure 6.9.

This should correspond, then, to the angle subtended by a half-plane that the Dirac string swept through as it went from the positive z to negative z axis.

From above, I hope that $\Omega_C = -2\phi$. First, suppose that I swept the Dirac string from the –z axis to the +z axis via the negative xy-plane. Further, suppose that ϕ runs from negative to positive pi, with zero corresponding to the positive z-axis. At zero, the plane swept by the string subtends none of my vision, appearing as a straight line. At $\frac{\pi}{2}$, it subtends exactly one-quarter of my vision corresponding to $\Omega_C = -2\frac{\pi}{2} = -\pi$. Finally, when I lay very near to but not on the plane the string sweeps through, it subtends fully half my vision for $\Omega_C = -2\pi = -2\pi$. Thus, this gauge transformation appears to be the reason for the discrepancy in vector potential under spatial inversion.

(Ficenec and Teplitz Problem 8.1) Show that $\vec{B} = \frac{g\hat{r}}{r^2}$ follows from

$$\partial_\mu \tilde{F}^{\mu\nu}(x) = \frac{-4\pi}{c}\sum_j g_j \int ds \frac{dz_j^\mu}{ds}\delta^4(x - z_j).$$

Suppose the magnetic monopole sits stationary at the origin. Then, for the spatial coordinates cooresponding to $\mu = 0$ I have:

$$-\vec{\nabla} \cdot \vec{B} = \frac{-4\pi}{c} g \int r \sin\theta dr d\theta d\phi \frac{dz^0}{dt} \frac{\delta(R-r)}{4\pi r^2} = \frac{-4\pi}{c} g \int r \sin\theta dr d\theta d\phi \frac{dz^0}{dt} \frac{\delta(R-r)}{4\pi r^2}$$
$$= \frac{-4\pi g}{Rc} \frac{dz^0}{dt}$$

so

$$-\vec{\nabla} \cdot \vec{B} = \frac{-4\pi g}{Rc} \frac{dz^0}{dt}$$
$$\vec{\nabla} \cdot \vec{B} = \frac{4\pi g}{Rc} \frac{dz^0}{dt} = \frac{g}{R}$$

But the form of the spherical del operator says that:

$$\vec{\nabla} \cdot \vec{B} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 B_r \right)$$

So inverting this I see that evidently

$$\vec{B} = \frac{g\hat{r}}{r^2} .$$

(Ficenec and Teplitz Problem 8.2) For a monopole of charge g at the origin and a positron of charge $+e$ at the point $\hat{z}e$ evaluate the angular momentu of the field explicitly.

$\vec{L}_F = \frac{1}{4\pi c} \int r' \times (\vec{E} \times \vec{B}) d^3 r'$ gives the angular momentum of the field. *Important.*

Allow me to shift the origin to be at the midpoint between the two charges, as the symmetry is likely to make life somewhat easier.

It is immediately clear that the angular momentum of the field will be entirely about the z-axis. Further, it is clear that there is rotational symmetry about the z-axis. Now I proceed in cylindrical coordinates with the shifted origin

$$\vec{E}\times\vec{B} = \frac{ge}{\left(r^2+\left(\frac{a}{2}+z\right)^2\right)^{\frac{3}{2}}\left(r^2+\left(\frac{a}{2}-z\right)^2\right)^{\frac{3}{2}}}\left(r\hat{r}+\left(z-\frac{a}{2}\right)\hat{z}\right)\times\left(r\hat{r}+\left(z+\frac{a}{2}\right)\hat{z}\right)$$

$$= \frac{ge}{\left(r^2+\left(\frac{a}{2}+z\right)^2\right)^{\frac{3}{2}}\left(r^2+\left(\frac{a}{2}-z\right)^2\right)^{\frac{3}{2}}}\left(-r\left(z+\frac{a}{2}\right)+r\left(z-\frac{a}{2}\right)\right)\hat{\theta}$$

$$= \frac{-rage}{\left(r^2+\left(\frac{a}{2}+z\right)^2\right)^{\frac{3}{2}}\left(r^2+\left(\frac{a}{2}-z\right)^2\right)^{\frac{3}{2}}}\hat{\theta}$$

Then,

$$\vec{r}'\times\left(\vec{E}\times\vec{B}\right) = \left(r\hat{r}+z\hat{z}\right)\times\frac{-rage}{\left(r^2+\left(\frac{a}{2}+z\right)^2\right)^{\frac{3}{2}}\left(r^2+\left(\frac{a}{2}-z\right)^2\right)^{\frac{3}{2}}}\hat{\theta}$$

$$= \frac{r^2age}{\left(r^2+\left(\frac{a}{2}+z\right)^2\right)^{\frac{3}{2}}\left(r^2+\left(\frac{a}{2}-z\right)^2\right)^{\frac{3}{2}}}\hat{z}$$

Integrating this, then, I get:

$$\vec{r}'\times\left(\vec{E}\times\vec{B}\right) = \left(r\hat{r}+z\hat{z}\right)\times\frac{-rage}{\left(r^2+\left(\frac{a}{2}+z\right)^2\right)^{\frac{3}{2}}\left(r^2+\left(\frac{a}{2}-z\right)^2\right)^{\frac{3}{2}}}\hat{\theta}$$

$$= \frac{r^2age}{\left(r^2+\left(\frac{a}{2}+z\right)^2\right)^{\frac{3}{2}}\left(r^2+\left(\frac{a}{2}-z\right)^2\right)^{\frac{3}{2}}}\hat{z}$$

Integrating,

$$\int_{0}^{\infty}\int_{-\infty}^{\infty} 2\pi r dr dz \frac{r^2 age}{\left(r^2+\left(\frac{a}{2}+z\right)^2\right)^{\frac{3}{2}}\left(r^2+\left(\frac{a}{2}-z\right)^2\right)^{\frac{3}{2}}}\hat{z}$$

$$= \frac{2\pi\mu_0 ge}{\varepsilon_0 a}\hat{z}\int_{-\frac{a}{2}}^{\frac{a}{2}} dz = 2\pi ge\hat{z}$$

I have performed the integral over the radial component in Mathematica. Note by symmetry that the region of space not between the two charges has opposing contribution to angular momentum. Thus, the integral over z collapses to just the region between the two charges.

Then,

$$\vec{L}_F = \frac{1}{4\pi c}\int r'\times\left(\vec{E}\times\vec{B}\right)d^3r' = \frac{ge}{2c}\hat{z}$$

Amazingly, this is independent of the separation of the two charges! If we believe that angular momentum is universally quantized in units of h-bar, this then implies the quantization of electric charge if a monopole exists.

(Ficenec and Teplitz Problem 8.3)

a) **Show that the solutions**

$$\vec{A}^a = \frac{g}{r\sin\theta}\left(1-\cos\theta\right)\hat{\phi}$$

$$\vec{A}^b = -\frac{g}{r\sin\theta}\left(1+\cos\theta\right)\hat{\phi}$$

are special cases of

$$\vec{A}_{\hat{n}} = -\frac{g}{r}\frac{\hat{n}\times\hat{r}}{1-\hat{n}\cdot\hat{r}}$$

where $\hat{n}$ is a unit vector.

$\vec{A}^b$ is simply the above formula with $\hat{n} = -\hat{k}$. The result is immediate upon substitution of $\hat{n}\times\hat{r} \rightarrow -\hat{k}\times\hat{r} = -\sin\theta\hat{\phi}$ and $\hat{n}\cdot\hat{r} \rightarrow -\hat{k}\cdot\hat{r} = -\cos\theta$ and the use of a trig identity.

$\vec{A}^a$ can only come from the above formula with $\hat{n} = \hat{k}$. Substitution similar to the one above and a trig identity quickly confirms this.

b) For a vector potential of this form, try to find $\Lambda_{\hat{n}\hat{m}}(\vec{r})$ such that $\vec{A}_{\hat{n}} - \vec{A}_{\hat{m}} = \vec{\nabla}\Lambda_{\hat{n}\hat{m}}(\vec{r})$.

An earlier Jackson problem makes it clear that this function should be equal to the amount of solid angle subtended by a plane in space with boundaries being the line pointing from the magnetic monopole out to infinity along $\hat{n}$ and the same along $\hat{m}$ as viewed from the point $\vec{r}$ (possibly within a sign).

(Ficenec and Teplitz Problem 8.4) Study the substitution

$$H = \frac{p^2}{2m} + V(r) \rightarrow \frac{1}{2m}\left(\vec{p} - \frac{e\vec{A}}{c}\right)^2 + V(\vec{r}) + e\phi(\vec{r})$$

in the classical case by writing out Hamilton's equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

for

$$H = \frac{p^2}{2m} + V(r)$$

and

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e\vec{A}}{c}\right)^2 + V(\vec{r}) + e\phi(\vec{r})$$

where $V(r)$ refers to all potential energy contributions other than electromagnetic. Show in particular for the second case that

$$m\ddot{q}_i = -\partial_i V + eE_i + \left(\frac{e}{c}\right)\varepsilon_{ijk}\dot{q}_j B_k$$

Hint: You will need to use

$$\dot{A}_i = \frac{\partial A_i}{\partial t} + \frac{dx_j}{dt}\frac{\partial A_i}{\partial x_j} = \frac{\partial A_i}{\partial t} + \dot{q}_j \partial_j A_i$$

The argument for this substitution stems from the substitution of a free Lagrangian plus an interaction Lagrangian for the total Lagrangian. The interaction Lagrangian can only be a dot between position and the vector potential as it must be both translationally and Lorentz invariant. The conjugate momentum, then, which is just be differentiation with respect to position coordinates of the Lagrangian, is as above.

Starting with the first Hamiltonian,

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -\partial_i V(r) \quad \dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m}$$

Neither of these comes as any surprise.

Now with the second Hamiltonian I have:

$$\dot{p}_i = -\partial_i H = \frac{e}{mc}\left(p_j - \frac{eA_j}{c}\right)\partial_i A_j - \partial_i V(r) - e\partial_i \phi(r)$$

$$-\partial_i \phi(r) - \frac{1}{c}\frac{\partial A_i}{\partial t} \to E_i \qquad \textbf{(8.4.1)}$$

$$= \frac{e}{mc}\left(p_j - \frac{eA_j}{c}\right)\partial_i A_j - \partial_i V(r) + eE_i + \frac{e}{c}\frac{\partial A_i}{\partial t}$$

I will return to this expression momentarily.

Using the other Hamilton's equation:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{1}{m}\left(p_i - \frac{eA_i}{c}\right) \qquad \textbf{(8.4.2)}$$

Then, substituting into the original equation (8.4.1),

$$\dot{p}_i = \frac{e}{c}\dot{q}_j \partial_i A_j - \partial_i V(r) + eE_i + \frac{e}{c}\frac{\partial A_i}{\partial t} \qquad \textbf{(8.4.3)}$$

Finally, I certainly have that:

$$\ddot{q}_i = \frac{\partial H}{\partial p_i} = \frac{1}{m}\left(\dot{p}_i - \frac{e\dot{A}_i}{c}\right)$$

$$m\ddot{q}_i + \frac{e\dot{A}_i}{c} = \dot{p}_i \qquad \textbf{(8.4.4)}$$

$$m\ddot{q}_i + \frac{e}{c}\left(\frac{\partial A_i}{\partial t} + \dot{q}_j \partial_j A_i\right) = \dot{p}_i$$

Substituting (8.4.3) into (8.4.4), I have:

$$m\ddot{q}_i + \frac{e}{c}\dot{q}_j \partial_j A_i = \frac{e}{c}\dot{q}_j \partial_i A_j - \partial_i V(r) + eE_i$$

$$m\ddot{q}_i = \frac{e}{c}\dot{q}_j\left(\partial_i A_j - \partial_j A_i\right) - \partial_i V(r) + eE_i = \frac{e}{c}\dot{q}_j \varepsilon_{ijk}\partial_i A_j - \partial_i V(r) + eE_i$$

$$= \frac{e}{c}\dot{q}_j \varepsilon_{ijk}\partial_i A_j - \partial_i V(r) + eE_i = \frac{e}{c}\dot{q}_j \varepsilon_{ijk} B_k - \partial_i V(r) + eE_i$$

The final line above utilizes a property of the Levi-Civita tensor.

(Ficenec and Teplitz Problem 8.5) Verify by direct substitution that

$$\psi(\vec{r}) = \exp\left\{ i\left[\vec{k}\cdot\vec{r} + \frac{e}{\hbar c}\int_{\vec{r}_o}^{\vec{r}} \vec{A}(\vec{r}\,')\cdot d\vec{r}\,' \right]\right\}$$

is a solution of

$$\frac{1}{2m}\left(\vec{p} - \frac{e\vec{A}}{c}\right)^2\psi(\vec{r}) = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e\vec{A}}{c}\right)^2\psi(\vec{r}) = E\,\psi(\vec{r})$$

with

$$\frac{\hbar^2 k^2}{2m} = E$$

$$\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2\psi(\vec{r})$$

$$= \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2\psi(\vec{r})$$

$$= \frac{1}{2m}\left(-\hbar^2\nabla^2 + i\hbar\frac{e}{c}\left(\vec{\nabla}\cdot\vec{A}\right) + i\hbar\frac{e}{c}\left(\vec{A}\cdot\vec{\nabla}\right) + \frac{e^2}{c^2}\vec{A}\cdot\vec{A}\right)\psi(\vec{r})$$

$$Note: \vec{\nabla}\psi(\vec{r}) \rightarrow \left(i\vec{k} + \frac{ie}{\hbar c}\vec{A}\right)\psi(\vec{r})$$

$$= \frac{1}{2m}\left(\hbar^2\left(k^2 + 2\frac{e}{\hbar c}\vec{A}\cdot\vec{k} + \frac{e}{\hbar c}A^2\right) + i\hbar\frac{e}{c}\left(\vec{\nabla}\cdot\vec{A}\right) + i\hbar\frac{e}{c}\left(\vec{A}\cdot\vec{\nabla}\right) + \frac{e^2}{c^2}A^2\right)\psi(\vec{r})$$

$$= \frac{1}{2m}\left(\hbar^2\left(k^2 + 2\frac{e}{\hbar c}\vec{A}\cdot\vec{k} + \frac{e}{\hbar c}A^2\right) + i\hbar\frac{e}{c}\left[\vec{A}\cdot\nabla\psi(\vec{r}) + \psi(\vec{r})\vec{\nabla}\cdot\vec{A}\right] + i\hbar\frac{e}{c}\left(\vec{A}\cdot\vec{\nabla}\right)\psi(\vec{r}) + \frac{e^2}{c^2}A^2\right)$$

$$= \frac{1}{2m}\left(\hbar^2\left(k^2 + 2\frac{e}{\hbar c}\vec{A}\cdot\vec{k} + \frac{e}{\hbar c}A^2\right)\psi(\vec{r}) + 2i\hbar\frac{e}{c}\left(\vec{A}\cdot\vec{\nabla}\right)\psi(\vec{r}) + \frac{e^2}{c^2}A^2\psi(\vec{r})\right)$$

$$= \frac{1}{2m}\left(\hbar^2\left(k^2 + 2\frac{e}{\hbar c}\vec{A}\cdot\vec{k} + \frac{e}{\hbar c}A^2\right) + 2i\hbar\frac{e}{c}\vec{A}\cdot\left(i\vec{k} + \frac{ie}{\hbar c}\vec{A}\right) + \frac{e^2}{c^2}A^2\right)\psi(\vec{r})$$

$$= \frac{1}{2m}\left(\hbar^2\left(k^2 + 2\frac{e}{\hbar c}\vec{A}\cdot\vec{k} + \frac{e}{\hbar c}A^2\right) - 2\hbar\frac{e}{c}\vec{A}\cdot\vec{k} - 2\frac{e^2}{c^2}A^2 + \frac{e^2}{c^2}A^2\right)\psi(\vec{r})$$

$$= \frac{\hbar^2 k^2}{2m}\psi(\vec{r}) = E\,\psi(\vec{r})$$

*** (Ficenec and Teplitz Problem 8.6) The classical current density of a particle** $\vec{j} = e\dot{\vec{q}}\,\delta(\vec{r}-\vec{q}) = \vec{j}(\vec{r})$ **is, in quantum mechanics,**

$$\vec{j}(\vec{r}) = -\frac{i\hbar}{2m}\left\{\left[\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}\right)\psi^*\right]\psi-\psi^*\left[\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}\right)\psi\right]\right\}.$$

Show that the space-dependent phase factor $\exp\left\{i\left[\frac{e}{\hbar c}\int_{\vec{r}_0}^{\vec{r}}\vec{A}(\vec{r}\,')\cdot d\vec{r}\,'\right]\right\} = e^{i\alpha(\vec{r})}$ **does not change** $\vec{j}(\vec{r})$**.**

Note that $\vec{\nabla}\exp\left\{i\left[\frac{e}{\hbar c}\int_{\vec{r}_0}^{\vec{r}}\vec{A}(\vec{r}\,')\cdot d\vec{r}\,'\right]\right\} = \vec{\nabla}e^{i\alpha(\vec{r})} = \frac{ie}{\hbar c}\vec{A}(\vec{r})e^{i\alpha(\vec{r})}$.

$$\vec{j}(\vec{r}) = -\frac{i\hbar}{2m}\left\{\left[\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}\right)\psi^*\right]\psi-\psi^*\left[\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}\right)\psi\right]\right\}$$

$$\vec{j}\,'(\vec{r}) = -\frac{i\hbar}{2m}\left\{\left[e^{-i\alpha(\vec{r})}\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}-\frac{ie}{\hbar c}\vec{A}\right)\psi^*\right]e^{i\alpha(\vec{r})}\psi-e^{-i\alpha(\vec{r})}\psi^*\left[e^{i\alpha(\vec{r})}\left(\vec{\nabla}+\frac{ie}{\hbar c}\vec{A}-\frac{ie}{\hbar c}\vec{A}\right)\psi\right]\right\}$$

$$= -\frac{i\hbar}{2m}\left\{\left[\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}-\frac{ie}{\hbar c}\vec{A}\right)\psi^*\right]\psi-\psi^*\left[\left(\vec{\nabla}+\frac{ie}{\hbar c}\vec{A}-\frac{ie}{\hbar c}\vec{A}\right)\psi\right]\right\}$$

$$= \vec{j}(\vec{r})-\frac{i\hbar}{2m}\left\{\left[\left(-\frac{ie}{\hbar c}\vec{A}\right)\psi^*\right]\psi-\psi^*\left[\frac{ie}{\hbar c}\vec{A}\psi\right]\right\}$$

But this doesn't seem to give the expected result.

I believe that there is an error in this problem and that in fact it should read:

$$\vec{j}(\vec{r}) = -\frac{i\hbar}{2m}\left\{\left[\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}\right)\psi^*\right]\psi-\left[\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}\right)\psi\right]\psi^*\right\}$$

If this had been the case I would have been left with:

$$= \vec{j}(\vec{r})-\frac{i\hbar}{2m}\left\{\left[\left(-\frac{ie}{\hbar c}\vec{A}\right)\right]\psi^*\psi-\left[\frac{ie}{\hbar c}\vec{A}\right]\psi\psi^*\right\} = \vec{j}(\vec{r})-\frac{e}{2mc}\vec{A}\left\{\psi^*\psi+\psi\psi^*\right\}$$

$$= \vec{j}(\vec{r})-\frac{e}{2mc}\vec{A}\left\{\psi^*,\psi\right\} = \vec{j}(\vec{r})$$

This is correct for Fermionic wave-functions, as in electrons and protons.

(Ficenec and Teplitz Problem 8.7) Consider a magnetic monopole somewhere in space. Follow the construction of section 2. For every r **choose regions** $X_{\alpha(r)}(\theta,\phi)$ **and** $Y_{\beta(r)}(\theta,\phi)$ **such that the overlap region circles the shaded areas in figures 2 and 3. This then shows in the Wu-Yang picture quantization of charges from the existence of one monopole.**

Choose a magnetic monopole located at the origin, and spherical coordinates. Angles are expressed in terms of θ with respect to the z-axis.

I choose $\beta = \frac{\pi}{4}$ and $\alpha = \frac{3\pi}{4}$.

They overlap in the region $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$.

Now I may choose, as in problem 8.3 above:

$$\vec{A}^a = \frac{g}{r\sin\theta}(1-\cos\theta)\hat{\phi}$$

$$\vec{A}^b = -\frac{g}{r\sin\theta}(1+\cos\theta)\hat{\phi}$$

These are valid in regions α and β, respectively. Note that

$$\vec{A}^a - \vec{A}^b = \frac{g}{r\sin\theta}\hat{\phi}$$

Now I have that

$$\alpha_c = \frac{e}{\hbar c}\left[\oint \vec{A}^\alpha \cdot d\vec{r} - \oint \vec{A}^b \cdot d\vec{r}\right] = \frac{e}{\hbar c}\oint \frac{2g}{r\sin\theta}\hat{\phi}\cdot d\vec{r} = 4\pi\frac{eg}{\hbar c}.$$

And for this to be not detectable, I need:

$$\alpha_c = 4\pi\frac{eg}{\hbar c} = 2\pi n$$

$$2\frac{eg}{\hbar c} = n$$

This is the Wu-Yang derivation of the quantization condition from the existence of a single monopole.

(Ficenec and Teplitz Problem 8.8) Prove that $\int_R \vec{\nabla}\times\vec{A}\,d^3r = \int_{\partial R}\hat{n}\times\vec{A}\,d^2r$ **.**

Counting repeated indices as summation,

$$\int_R \vec{\nabla}\times\vec{A}\,d^3r = \int_R \varepsilon_{ijk}\partial_i A_k\,d^3r = \int_R \partial_i\cdot\left[\varepsilon_{ijk}A_k\right]d^3r = \int_{\partial R}\left[\varepsilon_{ijk}A_k\right]\cdot n_i\,d^2r = \int_{\partial R}\hat{n}\times\vec{A}\,d^2r$$

(Ficenec and Teplitz Problem 8.9) (J. A. Jacobs) For geometries other than a sphere containing all of the distribution, show that $\vec{X}_J$ and $\vec{X}_{mp}$ are different than $\vec{X}_J = \frac{8\pi}{3}\vec{M}_J$ and $\vec{X}_{mp} = -\frac{4\pi}{3}\vec{M}_{mp}$ but, if $\vec{M}_J = \vec{M}_{mp}$, the equation $\vec{X}_J - \vec{X}_{mp} = 4\pi M$ remains true.

Let me first consider what, exactly, it means if $\vec{X}_J$ and $\vec{X}_{mp}$ were the same for any geometry of surface. This would mean that there is some scalar function such that $\vec{\nabla} \times \vec{A} = -\vec{\nabla}\phi_{mag}$ and $\nabla^2\phi_{mag} = \frac{\rho_{mag}}{\varepsilon_0}$ for sources, so that I get something akin to Gauss's Law when I enclose the distribution.

Evidently, finding such a function is a hopeless venture as if this were the case then:

$$\nabla^2\phi_{mag} = \vec{\nabla} \cdot \vec{\nabla}\phi_{mag} = -\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0 = \frac{\rho_{mag}}{\varepsilon_0}$$

This would constrain ϕ to be quite trivial—at most constant. With this, no interesting $\vec{\nabla} \times \vec{A}$ is possible so it is clear that this is not the type of quantity subject to a Gauss's Law type argument. Without any reason to believe that it is a sourced quantity subject to such a law, there is no reason to believe that $\vec{X}_J$ and $\vec{X}_{mp}$ should remain the same without regard to the shape of the surface containing the distribution. In fact, to do so would be to say that the average magnetic field in a distribution would be independent of the geometry of the choice of region.

Note that the mathematical expression above may seem counterintuitive, as we do indeed want to consider a magnetic charge density and a Gauss's Law for magnetic fields, but draw a distinction between this and allowing a Magnetic scalar and vector potential to exist. This is precisely what we want to avoid so as to prevent having to deal with multiple gauges.

$$\vec{M}_J = -\frac{1}{2c}\int_V \vec{r} \times \vec{J} d^3r = \vec{M}_{mp} = \int_V \vec{r}\rho_{mp}\left(\vec{r}\right)d^3r = ag\hat{k}$$

$$\vec{X}_J = \int_V \vec{\nabla} \times \vec{A}_J d^3r = \int_V \vec{\nabla} \times \frac{1}{c}\int_V \frac{\vec{J}\left(\vec{r}\,'\right)}{\left|\vec{r}\,'-\vec{r}\right|} d^3r' d^3r$$

$$\vec{X}_{mp} = \int_V \vec{\nabla} \times \left(\vec{A}_+ + \vec{A}_-\right)d^3r$$

There is probably some elegant way to solve this problem, but I cannot see it. Ideally, I would like to show a general property of the difference in the vector potetntials.

(Ficenec and Teplitz Problem 8.10) Show that $\vec{X}_J = \frac{8\pi}{3}\vec{M}_J$ follows from

$$\vec{X} = -\frac{R^2}{c}\int d^3r'\,\vec{J}(\vec{r}\,')\times\int_{r=R} d\Omega\frac{\hat{r}}{|\vec{r}\,'-\vec{r}|}$$

Note that $\int_{r=R} d\Omega\frac{\hat{r}}{|\vec{r}\,'-\vec{r}|} = \frac{4}{3}\pi\hat{r}'\left(\frac{r'}{r^2}\right)$ if $r' < R$, e.g. inside the region of interest. Then,

$$\vec{X} = -\frac{R^2}{c}\int d^3r'\,\vec{J}(\vec{r}\,')\times\frac{4}{3}\pi\hat{r}'\left(\frac{r'}{R^2}\right) = \frac{4\pi}{3c}\int d^3r'\,\vec{r}\,'\times\vec{J}(\vec{r}\,') = -\frac{8\pi}{3}\vec{M}_J$$

In light of this, the problem may be in error. Namely, it seems that the expression I was to show this result from should not have the minus sign.

(Ficenec and Teplitz Problem 8.11) (J. A. Jacobs) Show that in the Dirac string picture of monopoles, if the contribution to $\vec{X} = \int\vec{B}d^3r'$ from inside the strings are included, then $\vec{X}$ for the monopole distribution case is identical to $\vec{X}$ for the current distribution case.

This is relatively easy to show.

Let me try something a bit creative: I will take the Dirac string and run it from the positive monople to the negative monopole, so that the magnetic field is seen flowing into the negative charge and out at the positive charge. The "amount" of magnetic charge I will give to each monopole is then $4\pi\frac{g}{2}$ corresponding to the total magnetic flux integrated along a surface very close to the magnetic charge $\frac{g}{2}$.

The contribution of the string will be $\int_{-a}^{a} 4\pi\frac{g}{2}\hat{k}dr = 4\pi g a\hat{k} = 4\pi\vec{M}$, corresponding to exactly the difference observed.

(Ficenec and Teplitz Problem 8.12) Show that the average energy loss per unit distance of a magnetic charge g and mass M due to collisions with harmonically bound atomic electrons is:

$$\left(\frac{dE}{dx}\right)_g = \frac{4\pi NZe^2g^2}{m_ec^2}\left\{\ln\left[\frac{1.851\gamma^2m_ec^3\beta^2}{ge\langle\omega\rangle}\right]\right\}$$

Above, N is the number of atoms per unit volume, Z is the number of electrons per nucleus, $\langle\omega\rangle$ is the geometric mean of all of the harmonic frequencies of the electrons, β is the fraction of the speed of light that the particle travels at, and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

The quantization condition is $eg = n\frac{\hbar c}{2}$.

Let me first determine what the electric and magnetic fields must look like for a magnetic particle traveling at speed βc along the z-axis, in the frame of the stationary media.

$$\begin{aligned}
&B_z = \frac{\mu_0}{4\pi}\frac{g}{r^2}\hat{r}\cdot\hat{k} \quad E_z = 0 \\
&B_x = \gamma\frac{\mu_0}{4\pi}\frac{g}{r^2}\hat{r}\cdot\hat{i} \quad E_x = -\gamma\beta\frac{\mu_0}{4\pi}\frac{g}{r^2}\hat{r}\cdot\hat{j} \qquad \textbf{(8.12.1)} \\
&B_y = \gamma\frac{\mu_0}{4\pi}\frac{g}{r^2}\hat{r}\cdot\hat{j} \quad E_y = \gamma\beta\frac{\mu_0}{4\pi}\frac{g}{r^2}\hat{r}\cdot\hat{i}
\end{aligned}$$

Now let me cite a quantum-mechanical result from Bethe, who found that the energy loss in soft collisions for electrically charged particles is

$$\left(\frac{dE}{dx}\right)_q = \frac{4\pi NZz^2e^4}{m_ec^2}\left(\frac{1}{\beta^2}\right)\left\{\ln\left[\frac{1.123\gamma^2m_ec^3\beta^3}{ze^2\langle\omega\rangle}\right]-\frac{\beta^2}{2}\right\} \qquad \textbf{(8.12.2)}$$

The first substitution to make, as justified in (8.12.1) is $ze \to \beta g$, as I can see that the electric fields in the neighborhood of my fast monopole are thus comparable to the electric fields in the neighborhood of a fast electrically charged particle.

$$\left(\frac{dE}{dx}\right)_q = \frac{4\pi NZg^2e^2}{m_ec^2}\left\{\ln\left[\frac{1.123\gamma^2m_ec^3\beta^2}{ge\langle\omega\rangle}\right]-\frac{\beta^2}{2}\right\} \qquad \textbf{(8.12.2.1)}$$

Another serious difference is that the z-component of electric field, which would typically be the strongest contributor to this interaction, is absent. Thus, this large term vanishes.

$$\left(\frac{dE}{dx}\right)_q = \frac{4\pi NZg^2e^2}{m_ec^2}\left\{\ln\left[\frac{1.123\gamma^2m_ec^3\beta^2}{ge\langle\omega\rangle}\right]\right]\right\} \qquad \textbf{(8.12.2.2)}$$

The remaining difference is justified as such: The momentum of the monopole must be somewhat larger than that of an electric particle in order to produce the same transfer of energy due to its reduced fields.

(Ficenec and Teplitz Problem 8.15) Show that a pole-antipole pair with $g = \pm 137e$ created at a separation of 1 fermi will have $B \approx 30$ if their electromagnetic binding energy goes into photons.

Since this must entirely be provided in the center of mass system, let me calculate the binding energy corresponding to these particles at a separation of one Fermi. With straight substitution of the problem parameters into a magnetic potential analogous to the electric potential between two point charges, I have:

$$U = \frac{\mu_0}{4\pi}\frac{2g}{r} = 27.4GeV$$

$$\frac{27.4GeV}{m_p c^2} = \frac{27400MeV}{938MeV} = 29.211 = B$$

So apparently a great deal of energy would have to be provided to place these monopoles even a small distance apart.

(Ficenec and Teplitz Problem 8.16) Approximate the nucleus by assuming a uniform nucleon density and a radius $R = 1.2A^{1/3}$ Fermi. For a magnetic monopole at $r = 2R$ show that $W_A = -\left(\frac{\chi g}{2}\right)^2\left(0.242A^{-1/3}fm^{-4}\right)$. Also show that for the value of χ given above and $g = 137e$ this result becomes $W_A = -0.57A^{-1/3}MeV$. Compare this result with the value obtained using $W_A = -\frac{1}{2}A\chi\frac{g^2}{r^4}$.

In particular, I would like to use the equation

$$W_A = \frac{1}{2}\chi g^2 \sum_{i=1}^{A}\frac{1}{r_i^4}$$

Note that the author of the paper has omitted a minus sign from this expression.

for each nucleon to show that the equation in the problem statement is a valid approximation. First I need the nucleon density for this particular nucleus.

$$A = \rho_A \int_0^R 4\pi r^2 dr$$

$$\rho_A = \frac{3A}{4\pi R^3}$$

Now I just need to evaluate:

$$\sum_{i=1}^{A}\frac{1}{r_i^4}\to 2\pi\int_0^R\int_0^\pi\left[\frac{\rho_A}{\left[\left(r-r'\cos\theta\right)^2+\left(r'\sin\theta\right)^2\right]^2}\right]\sin\theta r'^2\,dr'$$

$$=2\pi\rho_A\int_0^R\int_0^\pi\left[\frac{\sin\theta r'^2}{\left(r^2-2rr'\cos\theta+r'^2\right)^2}\right]dr'$$

My justification for this is straightforward: I assume that the nucleons are uniformly distributed throughout the nucleus, and integrate over the expected weight of this sum from this point in the nucleus.

$$2\pi\rho_A\int_0^R\frac{2r'^2}{\left(r^2-r'^2\right)^2}dr'=2\pi\rho_A\left(\frac{R}{2\left(r^2-R^2\right)}-\frac{Arc\tanh\left[\frac{R}{r}\right]}{2r}\right)$$

Substitution gives:

$$=2\pi\frac{3A}{4\pi\left(1.2A^{1/3}\right)^3}\left(\frac{1.2A^{1/3}}{2\left(\left(2\cdot 1.2A^{1/3}\right)^2-\left(1.2A^{1/3}\right)^2\right)}-\frac{Arc\tanh\left[\frac{1}{2}\right]}{2\left(2\cdot 1.2A^{1/3}\right)}\right)$$

$$=\frac{3}{2\left(1.2\right)^3}\left(\frac{1}{6\cdot 1.2}-\frac{Arc\tanh\left[\frac{1}{2}\right]}{4\cdot 1.2}\right)A^{-1/3}=0.0212A^{-1/3}$$

In the full expression, then, I get:

$$W_A=\frac{1}{2}\chi g^2\left(0.0212A^{-1/3}\,fm^{-4}\right)$$

This is a far cry from the desired expression. I am not then clear on what approach the authors expected me to take, though my suspicion is that the author's solution is in error.

Taking $\chi\equiv 10^{-3}\ fm^3$ and $g=137e$ I get:

$$W_A=\frac{1}{2}\left(10^{-3}\right)\left(137^2\right)\left(0.0212\right)A^{-1/3}\,fm^{-1}e^2=\frac{1}{2}\left(10^{-3}\right)\left(137^2\right)\left(0.0212\right)\frac{e}{10^{15}}\left(\frac{C}{m}\right)e$$

Applying $\frac{1}{4\pi\varepsilon_0}\frac{C}{m} = V$, then, I have:

$$W_A = \frac{1}{2}\left(10^{-3}\right)\left(137^2\right)\left(0.0212\right)A^{-1/3}fm^{-1}e^2 = \frac{1}{2}\left(10^{-3}\right)\left(137^2\right)\left(0.0212\right)\frac{e}{10^{-15}}9\cdot 10^3 A^{-1/3}MeV$$
$$= 0.286A^{-1/3}MeV$$

So it looks like in the end I only disagree with the author by a factor of 2.

Comparing this to

$$W_A = \frac{1}{2}A\chi\frac{g^2}{r^4} \to \frac{1}{2}10^{-3}A\frac{\left(137e\right)^2}{\left(1.2A^{1/3}\right)^4} = 13.514A^{-1/3}MeV$$

It appears that the value calculated from the more precise estimate is roughly 2% of this. Apparently, getting very close to the nucleus decreases binding energy due to the distributed nature of the polarization of the charges. Thus, it would seem that the monopole-polarization binding energy plateaus at some distance after which getting any closer does not help substantially.

(Ficenec and Teplitz Problem 8.17) Show that for the interaction between a spinless nucleus of charge Ze and a monopole with no spin and no electric charge but with magnetic charge g, the time independent Schrodinger equation becomes:

$$\frac{1}{2M}\left\{-\frac{\hbar^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\left[L^2-\left(\frac{Zeg}{c}\right)^2\right]\right\}\psi = E\psi$$

where $M \approx M_A$ is the reduced mass of the system, approximately the mass of the nucleus for massive monopoles.

Let me start with an earlier-derived result that, for a monopole-electric charge system:

$$\vec{L} = \vec{r}\times m\vec{v} - \frac{eg}{c}\hat{r} = \vec{r}\times\vec{p} - \frac{eg}{c}\hat{r}$$
$$L^2 = \left(\vec{r}\times\vec{p}\right)^2 + \left(\frac{eg}{c}\right)^2 - 2\frac{eg}{c}\hat{r}\cdot\left(\vec{r}\times\vec{p}\right) = \left(\vec{r}\times\vec{p}\right)^2 + \left(\frac{eg}{c}\right)^2 . \quad \textbf{(8.17.1)}$$

It is well-known that in a system of two particles, the Schrodinger equation is separable into two parts: one describing the motion interior to the two-particle system and one describing the center-of-mass of the two-particle system. Ignoring the center of mass motion, then, and considering only the internal energy I have:

$$\frac{\hbar^2}{2M}\nabla^2\psi = E\psi \quad \textbf{(8.17.2)}$$

In spherical coordinates, this is:

$$=-\frac{\hbar^2}{2M}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\right\}\psi=E\psi \quad \textbf{(8.17.3)}$$

However, the quantum mechanical angular momentum operator is:

$$L^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \qquad \textbf{(8.17.4)}$$

Apparently, from (8.17.1), I must include the correction:

$$L^2-\left(\frac{eg}{c}\right)^2=(\vec{r}\times\vec{p})^2=(8.17.4) \qquad \textbf{(8.17.5)}$$

So that I get from (8.17.3)

$$=-\frac{1}{2M}\left\{\frac{\hbar^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\left(L^2-\left(\frac{Zeg}{c}\right)^2\right)\right\}\psi=E\psi$$

Here I have included the Z, as the charge of the nucleus in question is Ze. The fact that the reduced mass becomes approximately the mass of the nucleus is due to the fact that for heavy monopoles, the nucleus will be seen orbiting the monopole!

(Ficenec and Teplitz Problem 8.18) Show for the case of Hydrogen that $(r)_{\min}=1.06\dot{A}$ and $(W_a)_{\min}=-1.70eV$. Take $l_z=0, s_z=\frac{1}{2}, g=137e, \rho=0.53\dot{A}$.

Then,

$$P=\sum_{i=1}^{Z}\left(l_{z,i}+2s_{z,i}\right)=1.$$

$$D=\sum_{i=1}^{Z}\rho_i^2=0.2809\dot{A}^2$$

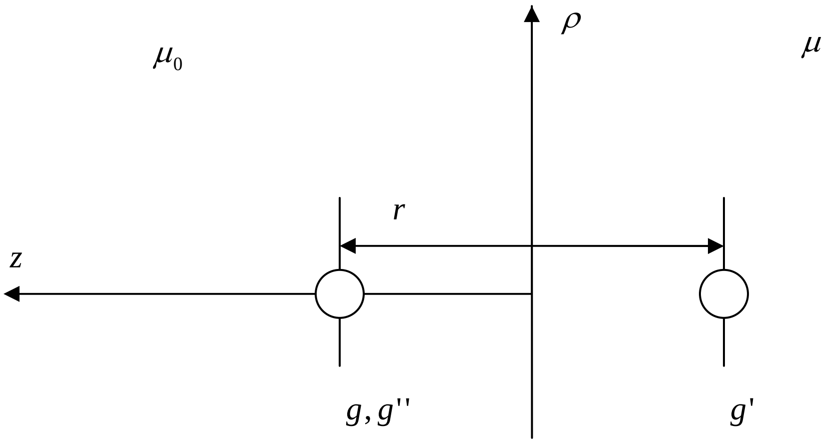
Being careful about my units, I get:

$$(r)_{\min}=\sqrt{\frac{4eg}{\hbar c}\frac{D}{P}}\rightarrow\sqrt{\frac{1}{4\pi\varepsilon_0}\frac{4(e\quad C)\left(g\quad\frac{N}{T}\right)}{(\hbar c\quad Jm)}\frac{D}{P}}=1.06\dot{A}$$

and

$$(W_a)_{\min}=-\frac{\hbar^2}{8m_eD}\left(1-\frac{P}{2}\right)\rightarrow-\frac{(\hbar\quad eVs)^2}{8(m_e\quad MeV)D}\left(c\quad\frac{m}{s}\right)^2\left(1-\frac{1}{2}\right)=-1.70\quad eV$$

(Ficenec and Teplitz Problem 8.19) Use the method of images to show that f or a semi-infinite slab with $\mu = 15$ a monopole will experience a force $\frac{7}{8}\frac{g^2}{(2r)^2}$ a distance r away from the slab. Therefore, the corrections to the ferromagnetic binding case of $F = \frac{g^2}{(2r)^2}$ are rather small even at modest values of μ.



I expect the "image magnetic charge" to be along the line perpendicular to the surface of the slab and intersecting my magnetic charge. Further, I expect that at the surface magnetic fields should be perpendicular to the surface, otherwise magnetic 'current' would be tempted to move along the surface.

Based on this, then, I first wish to find the strength of my image charge so that two things hold:

One, that there should be no magnetic charge anywhere inside the dielectric.

Two, that there should be no magnetic charge anywhere on the dielectric-surface.

As before, I would like to define a "magnetic scalar potential" concerning the image charge in the slab at a point symmetric to the magnetic monopole:

$$\phi_{mag}(z > 0) = \frac{\mu_0}{4\pi}\frac{g}{\left((z-r)^2 + \rho^2\right)^{\frac{1}{2}}} + \frac{\mu_0}{4\pi}\frac{g'}{\left((z+r)^2 + \rho^2\right)^{\frac{1}{2}}}$$

However, inside the slab I need to ensure that there are no magnetic monopoles, so I will place a second image charge at the same place as the original charge g:

$$\phi_{mag}(z<0) = \frac{\mu\mu_0}{4\pi}\frac{g''}{\left((z-r)^2+\rho^2\right)^{\frac{1}{2}}}$$

Now at the surface I need to constrain that there is no monopole charge. Namely, that

$$\frac{1}{\mu_0}\vec{B}_{z>0}\cdot\hat{z} = \frac{1}{\mu\mu_0}\vec{B}_{z<0}\cdot\hat{z}$$
$$\vec{B}_{z>0}\cdot\hat{\rho} = \vec{B}_{z<0}\cdot\hat{\rho}$$

at the surface.

Then, with $\vec{B} = -\vec{\nabla}\phi_{mag}$, I get:

$$\vec{B}_{z>0}(0) = \frac{\mu_0}{4\pi}\frac{g}{\left(r^2+\rho^2\right)^{\frac{3}{2}}}\left(-r\hat{k}+\rho\hat{\rho}\right)+\frac{\mu_0}{4\pi}\frac{g'}{\left(r^2+\rho^2\right)^{\frac{3}{2}}}\left(r\hat{k}+\rho\hat{\rho}\right)$$
$$\vec{B}_{z<0}(0) = \frac{\mu\mu_0}{4\pi}\frac{rg''}{\left(r^2+\rho^2\right)^{\frac{3}{2}}}\left(-r\hat{k}+\rho\hat{\rho}\right)$$

The surface conditions then indicate that:

$$g - g' = g''$$
$$g + g' = \mu g''$$

Solving this system given g, I have:

$$\frac{2}{(1+\mu)}g = g''$$
$$g' = \frac{(\mu-1)}{(1+\mu)}g$$

Here, then, I have $g' = \frac{14}{16}g = \frac{7}{8}g$. Straight substitution of this image into the Coulomb force law for monopoles, I get $F = \frac{gg'}{R^2} = \frac{7}{8}\frac{g^2}{(2r)^2}$.

(Ficenec and Teplitz Problem 8.20) Obtain $t \cong \frac{2\cdot 10^{20}}{T^2}N(T)^{-1/2}$.

Using, for the early universe, the expansion rate and density of particles:

$$\dot{R}^2 \cong \frac{8\pi G}{3}\rho R^2 \quad \rho \cong N(T)\frac{(kT)^4}{(\hbar c)^3 c^2}$$

By substitution I have:

$$\dot{R}^2 = \frac{8\pi G}{3} N(T)\frac{(kT)^4}{(\hbar c)^3 c^2} R^2$$

For an adiabatic expansion, I may let $R \propto \frac{1}{T} \rightarrow R = \frac{\alpha}{T}$

$$\dot{R}^2 = \frac{8\pi G}{3} N(T)\frac{k^4 T^2}{(\hbar c)^3 c^2}\alpha^2$$

$$\dot{R} = \sqrt{\frac{8\pi G}{3\hbar c}\frac{k^2}{\hbar c^2}} N(T)^{1/2}\alpha T$$

Then,

$$\dot{R}^2 = \frac{8\pi G}{3} N(T)\frac{k^4 T^2}{(\hbar c)^3 c^2}\alpha^2$$

$$\int \dot{R}dt = \int \sqrt{\frac{8\pi G}{3\hbar c}\frac{k^2}{\hbar c^2}} N(T)^{1/2} T^2 R dt$$

And by the crudest imaginable math,

$$R \cong \frac{1}{2}\sqrt{\frac{8\pi G}{3\hbar c}\frac{k^2}{\hbar c^2}} N(T)^{1/2} T^2 R t$$

$$t \cong \frac{1}{2}\left[\sqrt{\frac{8\pi G}{3\hbar c}\frac{k^2}{\hbar c^2}}\right]^{-1}\frac{N(T)^{-1/2}}{T^2}$$

$$t \cong 1.873 \cdot 10^{20}\, sK^2\, \frac{N(T)^{-1/2}}{T^2}$$

Above I use G in CGS units, h-bar in units of erg s, k in units of erg / K and c in units of cm / s.

(Ficenec and Teplitz Problem 8.21) Solve $\dot{R}^2 \cong \frac{8\pi G}{3}\rho R^2$ for $R(t)$ in the case when $\rho(R) = \rho(R_0)\left(\frac{R_0}{R}\right)^3$, where R_0 is the scale size at some t_0. Show for this case $t \propto T^{-3/2}$.

$$\dot{R}^2 \cong \frac{8\pi G}{3}\rho R^2 = \left(\frac{8\pi G\rho(R_0)R_0^3}{3}\right)\frac{1}{R}$$

Solving,

$$\dot{R} \cong \frac{8\pi G}{3}\rho R^2 = \sqrt{\left(\frac{8\pi G\rho(R_0)R_0^3}{3}\right)}\frac{1}{R^{1/2}}$$

$$\alpha \equiv \sqrt{\left(\frac{8\pi G\rho(R_0)R_0^3}{3}\right)}$$

$$R(t) = \frac{1}{4}(\alpha t + C)^2$$

So, if $R(t_0) = R_0$, then $R_0 = \frac{1}{4}(\alpha t_0 + C)^2$ and $C = 2\sqrt{R_0} - \alpha t_0$.

Then,

$$R(t) = \left(\frac{\alpha}{2}(t - t_0) + \sqrt{R_0}\right)^2$$

$$t = \frac{1}{\alpha}\left(\sqrt{R} - \sqrt{R_0}\right) + t_0 .$$

However, if

$R \propto \frac{1}{T}$ in the adiabatic case, then I have:

$$t \propto \sqrt{R} - \sqrt{R_0} \propto T^{-1/2} - T_0^{-1/2} \propto \partial_T T^{-1/2}\Big|_{T_0} \propto T^{-3/2}$$

(Ficenec and Teplitz Problem 8.22) Show that the solution of $\frac{d(n_M n_\gamma)}{dt} \cong \left\{\frac{-g^2c^2\hbar}{(kT)^2}\right\}\left(\frac{n_M}{n_\gamma}\right)^2 n_\gamma$ **gives** $\frac{n_M}{n_\gamma} \cong 10^{-10}$ **at** $kT \cong 10^{10}\,GeV$ **for any starting value of** $\frac{n_M}{n_\gamma} > 10^{-10}$ **at any** $kT > 10^{10}\,GeV$ **.**

Using $dt = \frac{dT}{T^3}$, I get:

$$\frac{d(n_M n_\gamma)}{d(kT)} = \left\{\frac{-g^2c^2\hbar k^2}{n_\gamma^3}\right\}\frac{(n_M n_\gamma)^2}{(kT)^5}$$

The solution to this is then:

$$\alpha \equiv \frac{g^2 c^2 \hbar k^2}{n_\gamma^3} = \frac{1}{n_\gamma^3}$$

$$n_M n_\gamma (kT) = -\frac{4(kT)^4}{\alpha + 4C(kT)^4}$$

Now if I would like to specify

$$n_M n_\gamma (kT_0) = -\frac{4(kT_0)^4}{\alpha + 4C(kT_0)^4}$$

$$C = -\frac{\alpha n_M n_\gamma (kT_0) + 4(kT_0)^4}{4 n_M n_\gamma (kT_0)(kT_0)^4}$$

Then,

$$n_M n_\gamma (kT) = \left[\frac{4}{\frac{\alpha n_M n_\gamma (kT_0) + 4(kT_0)^4}{n_M n_\gamma (kT_0)(kT_0)^4} - \alpha (kT)^{-4}} \right]$$

I would proceed by finding the necessary orders of magnitude, assuming that the number density of photons is constant, in order to make this give the desired result. However, it is not immediately clear to me how $\alpha \equiv \frac{g^2 c^2 \hbar k^2}{n_\gamma^3} \propto J^4$ without severe mangling of units.