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Griffiths 4.44

Griffiths 4.44) Suppose two spin-half particles are known to be in the singlet configuration. Let $S_a^{(1)}$ be the component of the spin angular momentum of particle number 1 in the direction defined by the unit vector $\hat{a}$. Similarly, let $S_b^{(2)}$ be the component of 2's angular momentum in the direction $\hat{b}$. Show that:

$$\left\langle S_a^{(1)} S_b^{(2)} \right\rangle = -\frac{\hbar^2}{4}\cos\theta$$

Where θ is the angle between $\hat{a}$ and $\hat{b}$.

It is important to note that what is meant by the "singlet" state corresponds to the Clebsch-Gordan state $\left|0 \quad 0\right\rangle$, or $\frac{1}{\sqrt{2}}\left|\uparrow_1\right\rangle\left|\downarrow_2\right\rangle - \frac{1}{\sqrt{2}}\left|\downarrow_1\right\rangle\left|\uparrow_2\right\rangle$.

There are, as always, many ways to do this. However, if I were an undergraduate seeing this for the first time, it would be really important to see the implications of a "vector" operator, and that it works mathematically just the same as any other vector and that moreover there is nothing new and magical at work here.

Let's consider what is meant by the operators $S_a^{(1)}$ and $S_b^{(2)}$. $S_a^{(1)}$ is the component of particle 1's spin in the $\hat{a}$ direction. Certainly, then, I can write particle 1's spin as:

$$\vec{S}^{(1)} \equiv S_x^{(1)}\hat{x} + S_y^{(1)}\hat{y} + S_z^{(1)}\hat{z} .$$

However, the component in the $\hat{a}$ direction is:

$$S_a^{(1)} = \vec{S}^{(1)} \cdot \hat{a} = S_x^{(1)}\hat{x}\cdot\hat{a} + S_y^{(1)}\hat{y}\cdot\hat{a} + S_z^{(1)}\hat{z}\cdot\hat{a} .$$

I could write something similar for $S_b^{(2)}$.

Now I'm left with:

$$
\begin{aligned}
&\left\langle S_a^{(1)} S_b^{(2)} \right\rangle \\
&= \left\langle \left(S_x^{(1)} \hat{x}\cdot\hat{a} + S_y^{(1)} \hat{y}\cdot\hat{a} + S_z^{(1)} \hat{z}\cdot\hat{a} \right)\left(S_x^{(2)} \hat{x}\cdot\hat{b} + S_y^{(2)} \hat{y}\cdot\hat{b} + S_z^{(2)} \hat{z}\cdot\hat{b} \right) \right\rangle \\
&= \left\langle S_x^{(1)} S_x^{(2)} \right\rangle (\hat{x}\cdot\hat{a})(\hat{x}\cdot\hat{b}) + \left\langle S_y^{(1)} S_y^{(2)} \right\rangle (\hat{y}\cdot\hat{a})(\hat{y}\cdot\hat{b}) + \left\langle S_z^{(1)} S_z^{(2)} \right\rangle (\hat{z}\cdot\hat{a})(\hat{z}\cdot\hat{b}) \\
&+ \left\langle S_x^{(1)} S_y^{(2)} \right\rangle (\hat{x}\cdot\hat{a})(\hat{y}\cdot\hat{b}) + \left\langle S_x^{(1)} S_z^{(2)} \right\rangle (\hat{x}\cdot\hat{a})(\hat{z}\cdot\hat{b}) + \left\langle S_y^{(1)} S_x^{(2)} \right\rangle (\hat{y}\cdot\hat{a})(\hat{x}\cdot\hat{b}) \\
&+ \left\langle S_y^{(1)} S_z^{(2)} \right\rangle (\hat{y}\cdot\hat{a})(\hat{z}\cdot\hat{b}) + \left\langle S_z^{(1)} S_x^{(2)} \right\rangle (\hat{z}\cdot\hat{a})(\hat{x}\cdot\hat{b}) + \left\langle S_z^{(1)} S_y^{(2)} \right\rangle (\hat{z}\cdot\hat{a})(\hat{y}\cdot\hat{b})
\end{aligned}
$$

What you see above looks terrible. It is pretty bad, but six of these will turn out to be zero. From looking at the answer, you should be able to guess which six. Clearly, $(\hat{x}\cdot\hat{a})(\hat{x}\cdot\hat{b}) + (\hat{y}\cdot\hat{a})(\hat{y}\cdot\hat{b}) + (\hat{z}\cdot\hat{a})(\hat{z}\cdot\hat{b}) = \hat{a}\cdot\hat{b} \equiv \cos\theta$. Now I expect that these coefficients $\left\langle S_x^{(1)} S_x^{(2)} \right\rangle, \left\langle S_y^{(1)} S_y^{(2)} \right\rangle, \left\langle S_z^{(1)} S_z^{(2)} \right\rangle$ will conspire to give me the rest of the solution, and I expect the other six to go to zero.

So what is the value of the operators $S_x^{(1)}, S_y^{(1)}, S_z^{(1)}, S_x^{(2)}, S_y^{(2)}, S_z^{(2)}$ acting on my states? I simply read these off of the Pauli Spin Matrices!

$$
\begin{aligned}
&S_x|\uparrow\rangle = \frac{\hbar}{2}|\downarrow\rangle \quad S_y|\uparrow\rangle = -\frac{i\hbar}{2}|\downarrow\rangle \quad S_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle \\
&S_x|\downarrow\rangle = \frac{\hbar}{2}|\uparrow\rangle \quad S_y|\downarrow\rangle = \frac{i\hbar}{2}|\uparrow\rangle \quad S_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle
\end{aligned}
$$

Further, the definition of "expectation value" here is:

$$
\begin{aligned}
&\langle Operator\rangle \to \left[\frac{1}{\sqrt{2}}\langle\uparrow_1|\langle\downarrow_2| - \frac{1}{\sqrt{2}}\langle\downarrow_1|\langle\uparrow_2|\right][Operator]\left[\frac{1}{\sqrt{2}}|\uparrow_1\rangle|\downarrow_2\rangle - \frac{1}{\sqrt{2}}|\downarrow_1\rangle|\uparrow_2\rangle\right] \\
&= \frac{1}{2}\left[\langle\uparrow_1|\langle\downarrow_2| - \langle\downarrow_1|\langle\uparrow_2|\right][Operator]\left[|\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle\right]
\end{aligned}
$$

Now simply applying the two formulas above to each case:

$$\left\langle S_x^{(1)}S_x^{(2)}\right\rangle = -\frac{1}{2}\left[\langle\uparrow_1|\langle\downarrow_2| - \langle\downarrow_1|\langle\uparrow_2|\right]\left[S_x^{(1)}S_x^{(2)}\right]\left[|\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle\right]$$

$$= \frac{1}{2}\left[\langle\uparrow_1|\langle\downarrow_2| - \langle\downarrow_1|\langle\uparrow_2|\right]\left[\frac{\hbar}{2}|\downarrow_1\rangle\frac{\hbar}{2}|\uparrow_2\rangle - \frac{\hbar}{2}|\uparrow_1\rangle\frac{\hbar}{2}|\downarrow_2\rangle\right]$$

$$= \frac{1}{2}\frac{\hbar^2}{4}\left[\langle\uparrow_1|\langle\downarrow_2| - \langle\downarrow_1|\langle\uparrow_2|\right]\left[|\downarrow_1\rangle|\uparrow_2\rangle - |\uparrow_1\rangle|\downarrow_2\rangle\right]$$

$$= \frac{1}{2}\frac{\hbar^2}{4}\left[-\langle\uparrow_1|\langle\downarrow_2\|\uparrow_1\rangle|\downarrow_2\rangle + \langle\downarrow_1|\langle\uparrow_2\|\uparrow_1\rangle|\downarrow_2\rangle + \langle\downarrow_1|\langle\uparrow_2\|\uparrow_1\rangle|\downarrow_2\rangle - \langle\downarrow_1|\langle\uparrow_2\|\downarrow_1\rangle|\uparrow_2\rangle\right]$$

$recall:$

$$\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$$

$$\langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 0$$

$$= \frac{1}{2}\frac{\hbar^2}{4}\left[-\langle\uparrow_1\|\uparrow_1\rangle\langle\downarrow_2\|\downarrow_2\rangle + \langle\downarrow_1\|\uparrow_1\rangle\langle\uparrow_2\|\downarrow_2\rangle + \langle\downarrow_1\|\uparrow_1\rangle\langle\uparrow_2\|\downarrow_2\rangle - \langle\downarrow_1\|\downarrow_1\rangle\langle\uparrow_2\|\uparrow_2\rangle\right]$$

$$= \frac{1}{2}\frac{\hbar^2}{4}\left[-1+0+0-1\right] = -\frac{\hbar^2}{4}$$

By following the same procedure with the formulas above, you will find:

$$\left\langle S_x^{(1)}S_x^{(2)}\right\rangle = \left\langle S_y^{(1)}S_y^{(2)}\right\rangle = \left\langle S_z^{(1)}S_z^{(2)}\right\rangle = -\frac{\hbar^2}{4}$$

$$\left\langle S_x^{(1)}S_y^{(2)}\right\rangle = \left\langle S_x^{(1)}S_z^{(2)}\right\rangle = \left\langle S_y^{(1)}S_x^{(2)}\right\rangle = \left\langle S_y^{(1)}S_z^{(2)}\right\rangle = \left\langle S_z^{(1)}S_x^{(2)}\right\rangle = \left\langle S_z^{(1)}S_y^{(2)}\right\rangle = 0$$

Then I have:

$$\left\langle S_a^{(1)}S_b^{(2)}\right\rangle = \left\langle S_x^{(1)}S_x^{(2)}\right\rangle(\hat{x}\cdot\hat{a})(\hat{x}\cdot\hat{b}) + \left\langle S_y^{(1)}S_y^{(2)}\right\rangle(\hat{y}\cdot\hat{a})(\hat{y}\cdot\hat{b}) + \left\langle S_z^{(1)}S_z^{(2)}\right\rangle(\hat{z}\cdot\hat{a})(\hat{z}\cdot\hat{b})$$

$$= -\frac{\hbar^2}{4}(\hat{x}\cdot\hat{a})(\hat{x}\cdot\hat{b}) - \frac{\hbar^2}{4}(\hat{y}\cdot\hat{a})(\hat{y}\cdot\hat{b}) - \frac{\hbar^2}{4}(\hat{z}\cdot\hat{a})(\hat{z}\cdot\hat{b}) = -\frac{\hbar^2}{4}(\hat{a}\cdot\hat{b}) = -\frac{\hbar^2}{4}\cos\theta$$

As you can see, there is no new math here. You only do things you already know how to do. You define the operator in terms of a vector (just the same as any other Cartesian vector), then you simply let your operators act on the states via the Pauli Spin Matrices. You get your expectation value by collapsing the original state against the result of this operation.