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Continuum Mechanics Homework 5

Note: My editor does not allow some necessary symbols. Let:

$\underline{R}$ **denote reals,** $\bar{x}$ **denote a position vector,** x **denote a position and time, and underlined symbols denote their a class.**

Gurtin 16.1) Consider a material body $\underline{B}$ **with constitutive class** $\underline{C}$**. A simple constraint for** $\underline{B}$ **is a function**

$$\gamma : Lin^{+} \to \underline{R}$$

such that each dynamical process $(x,T)\in \underline{C}$ **satisfies**

$$\gamma(F)=0$$

For such a material one generally lays down the following constraint axiom: The stress is determined by the motion only to within a stress N **that does no work in any motion consistent with the constraint. [The rate at which a stress** N **does work is given by the stress power per unit volume**

$$N\cdot D$$

(cf. page 111), where D **is the stretching.]**

We now make this idea precise. Let $\underline{D}$ **be the set of all possible stretching tensors; that is,** $\underline{D}$ **is the set of all tensors** D **with the following property: For some** C^2 **function** $F:\underline{R}\to Lin^{+}$ **consistent with** $\gamma(F)=0$**,** D **is the symmetric part of**

$$L=\dot{F}(t)F(t)^{-1}$$

at some fixed time t**. Let**

$$\Re=\underline{D}^{\perp};$$

that is,

$$\Re=\{N\in Sym \mid N\cdot D=0 \quad for \quad all \quad D\in \underline{D}\}.$$

Then the constraint axiom can be stated as follows: If $(x,T)\in\underline{C}$**, then so also does every dynamical process of the form** $(x,T+N)$ **with**

$$N(\bar{x},t) \in \Re$$

for all $(\bar{x},t)$ in the trajectory of x. We call $\Re$ the reaction space.

An incompressible material can be defined by the simple constraint

$$\gamma(F) = \det F - 1$$

Show that this constraint satisfies the constraint axiom if and only if the corresponding reaction space is the set of all tensors of the form $-\pi I$, $\pi \in \underline{R}$; i.e., if and only if the stress is determined by the motion at most to within an arbitrary pressure field. Show further that the constitutive assumptions of an ideal fluid are consistent with the constraint axiom.

First, let me determine the set of valid stretching tensors $\underline{D}$.

This set, as stated above, is the set of symmetric parts of L corresponding to valid deformation gradients F .

In this motion, then, my constraint $\gamma(F) = \det F - 1 = 0$ fixes

$$\det F = 1 .$$

However, since $\dot{F} = L_m F$ and $(\det \quad S)^{\bullet} = (\det \quad S) tr(\dot{S}S^{-1})$ for a smooth tensor-valued function S , then

$$(\det \quad F)^{\bullet} = (\det \quad F) tr(\dot{F}F^{-1}) = (\det \quad F)(tr \quad L_m)$$

Substituting,

$$(1)^{\bullet} = 0 = (\det \quad F)(tr \quad L_m) = tr \quad L_m$$

Now I see that $tr \quad L_m = 0$. Based on this, then, the set of valid stretching tensors must have $L = D + W$, with $tr \quad W = 0$ and $tr \quad L = 0$, so that $\underline{D} = \{D \in Sym \,|\, tr \quad D = 0\}$.

First, going in the if direction, I show that the reaction space of the form $N = -\pi I$ causes the constraint $\gamma(F) = \det F - 1$ to satisfy the constraint axiom. In this case, then, with $D \in \underline{D} = \{D \in Sym \,|\, tr \quad D = 0\}$,

$$N \cdot D = -\pi I \cdot D = -\pi \sum_{i=1}^{3} D_{ii} = -\pi(tr \quad D) = 0 \ .$$

Thus, I see that the reaction space $N = -\pi I$ causes the constraint $\gamma(F) = \det F - 1$ to satisfy the constraint axiom.

Next, going in the only if direction, I suppose that a reaction space $\Re$ is produced from the constraint $\gamma(F) = \det F - 1$ so as to satisfy the constraint axiom. Take an $N \in \Re$ and then decompose it as such:

$$N = -\pi I + N_0$$
$$\pi = -\frac{1}{3} tr \quad N_0$$

Now,

$$N \cdot D = (-\pi I + N_0) \cdot D = -\pi \sum_{i=1}^{3} D_{ii} + N_0 \cdot D = -\pi(tr \quad D) + N_0 \cdot D = N_0 \cdot D$$

Suppose $N_0 \in Sym$ has an non-zero off-diagonal element. If this is the case, then there exists a traceless symmetric matrix D with zeroes everywhere but at the corresponding elements, and $N_0 \cdot D \neq 0$ for this D. Thus, $N_0 \in Sym$ has only zero elements off-diagonal.

Suppose $N_0 \in Sym$ has non-zero on-diagonal elements (not all of N_0's on-diagonal elements are the same, by virtue of the decomposition above). If this is the case, then there exists a traceless D with elements such that $\sum_i N_{ii} D_{ii} \neq 0$ as evidenced by the fact that not all N_{ii} are the same. Thus, the on-diagonal elements of $N_0 \in Sym$ must be zero.

Since $N_0 \in Sym$ is thus zero everywhere, $N_0 = 0$ which indicates that

$$N = -\pi I + N_0 = -\pi I + 0 = -\pi I \,.$$

An ideal fluid is a material body consistent with the constitutive assumptions that (a) the class is the set of all isochoric, Eulerian dynamical systems and (b) that the density is constant. The process is Eulerian if the stress is a pressure, as in the isochoric case above. Thus I see that the ideal fluid's first assumption is consistent with the constraint axiom. By fixing the initial body's density as a constant, then, the fact that the motion is isochoric and so preserves the volume of each part of the body in motion affords the density no option to change, and so this constitutive assumption is consistent with the same constraint axiom; only the initial conditions must be set properly in order to ensure constant density under an isochoric motion.

Gurtin 19.2) Consider the ideal gas in equilibrium ($\dot{v} = 0$) under the gravitational body force

$$b = -\rho g e_3 \text{,}$$

where g **is the gravitational constant. Assume that**

$$\pi(x) = \pi_0 = \textbf{constant at } x_3 = 0 \text{.}$$

Determine the pressure distribution as a function of height x_3**.**

Recall that an ideal gas is an elastic fluid defined by a constitutive equation of the form:

$$\pi = \lambda \rho^{\gamma} \text{ with } \lambda > 0 \text{ and } \gamma > 1$$

One of the basic equations of flow is then:

$$-\,grad \quad \pi + b = \rho \dot{v}$$

However, in equilibrium $\dot{v} = 0$ and I have

$$-\,grad \quad \pi + b = 0$$

Since pressure is constant at $x_3 = 0$ and the gravitational force is entirely in the e_3 direction, I can assume from the symmetry of this problem that the density is a function of x_3 only, e.g. $\rho(x_3)$.

Then,

$$-\,grad \quad \lambda \rho(x_3)^{\gamma} - \rho(x_3) g e_3 = 0$$

$$-\lambda \gamma \rho(x_3)^{\gamma-1} \frac{\partial \rho}{\partial x_3} e_3 - \rho(x_3) g e_3 = 0$$

$$-\lambda \gamma \rho(x_3)^{\gamma-1} \frac{\partial \rho}{\partial x_3} - \rho(x_3) g = 0$$

$$\frac{\partial \rho}{\partial x_3} = -\rho(x_3)^{2-\gamma} \frac{g}{\lambda \gamma}$$

The solution to this differential equation is

$$\rho(x_3) = \left[(1-\gamma)\left(\frac{gx}{\lambda \gamma} - C \right) \right]^{\frac{1}{\gamma-1}}$$

Note that this solution can give negative densities after a particular distance, and so the solution would have to be re-evaluated at this limit. Then,

$$\pi = \lambda\rho^{\gamma} = \lambda\left[(1-\gamma)\left(\frac{gx}{\lambda\gamma} - C\right)\right]^{\frac{\gamma}{\gamma-1}}$$

Solving to eliminate the initial condition,

$$\pi_0 = \pi(0) = \lambda[(1-\gamma)(-C)]^{\frac{\gamma}{\gamma-1}}$$

$$C = \frac{1}{\gamma-1}\left(\frac{\pi_0}{\lambda}\right)^{\frac{\gamma-1}{\gamma}}$$

So, finally:

$$\pi(x) = \lambda\left[\frac{gx}{\lambda\gamma}(1-\gamma) + \left(\frac{\pi_0}{\lambda}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}}$$

Gurtin 21.1) Prove that the response of an elastic fluid is independent of the observer.

The constitutive class of elastic fluids is defined by having $T = -\hat{\pi}(\rho)I$, with $\hat{\pi}(\rho)$ a given function of density.

Let $\underline{C}$ be the constitutive class of an elastic fluid, let $(x,T) \in \underline{C}$ and let (x^*,T^*) be related to (x,T) by a change in observer. Then, I must show that (x^*,T^*) has $T^* = -\hat{\pi}(\rho)I$.

Then, $T^*(x^*,t) = QT(x,t)Q^T = Q[-\hat{\pi}(\rho)I]Q^T = -\hat{\pi}(\rho)QQ^T = -\hat{\pi}(\rho)I$.

Since x and x^* must necessarily correspond to the same material point, then the density is the same in both cases and $T^* = T = -\hat{\pi}(\rho)I$ and elastic fluids are independent of the observer.

Gurtin 22.1) A Reiner-Rivlin fluid is defined by the constitutive assumptions of the Newtonian fluid with the assumption of linearity removed. Use the First Representation Theorem for Isotropic Tensor Functions to show that the response of a Reiner-Rivlin fluid is independent of the observer if and only if the constitutive equation has the form

$$T = -\pi I + \alpha_0(I_D)D + \alpha_1(I_D)D^2$$

with $\alpha_0(I_D)$ and $\alpha_1(I_D)$ scalar functions of the list $I_D = \{0, \iota_2(D), \iota_3(D)\}$ of principal invariants of D.

The First Representation Theorem for Isotropic Tensor Functions says that any function $G : \underline{A} \to Sym$ with $\underline{A} \subset Sym$ is isotropic if and only if there exists scalar functions $\varphi_0, \varphi_1, \varphi_2 : I(\underline{A}) \to \underline{R}$ such that $G(A) = \varphi_0(I_A)I + \varphi_1(I_A)A + \varphi_2(I_A)A^2$ for all $A \subset \underline{A}$.

First, I need to find the transformation rule for D. Writing, as in Gurtin:
$x^* = f(x,t)$

with the identification $grad \quad f = Q$, one can then use:
$x^*(p,t) = q(t) + Q(t)[x(p,t) - o]$ for a material point p.

Now the derivative of this gives
$\dot{x}^*(p,t) = q(t) + Q(t)\dot{x}(p,t) + \dot{Q}(t)[x(p,t) - o]$

and taking $p = p^*(x^*, t)$:
$v^*(x^*, t) = \dot{q}(t) + Q(t)v(x,t) + \dot{Q}(t)(x - o)$.

Taking the gradient, then, we get:
$L(x,t) = grad \quad v$
$L^*(x^*, t)Q(t) = [grad \quad v^*]Q(t) = Q(t)L(x,t) + \dot{Q}(t)Q(t)^T$

But as shown in problem set 1, problem 3.6, $\dot{Q}(t)Q(t)^T$ is skew so that the change of observer becomes:
$D^*(x^*, t) = \frac{1}{2}(L^* + L^{*T}) = Q(t)D(x,t)Q(t)^T$.

The only constitutive assumptions of a Reiner-Rivlin fluid are that $T = -\pi I + \alpha_0(I_D)D + \alpha_1(I_D)D^2$ and that $\det \quad F = 1$ with initial condition that has constant density ρ_0. In this way, the motion is isochoric.

Note that with $D = \frac{1}{2}(L + L^T)$ and $\dot{F} = LF$ so that $\dot{F}F^{-1} = L$, D is a function of an F satisfying $\det \quad F = 1$ for a motion, so that the form of the stress tensor T does not give a constraint on F. This indicates that the First Representation Theorem only establishes a bijection between independence of observer and the form of the stress tensor and has no implication for the isochoric nature of F required for the other constitutive assumption of a Reiner-Rivlin fluid.

Now, since $D \subset Sym$, $T \subset Sym$ and the principal invariants I_D are given I have:
$D^*(x^*, t) = Q(t)D(x,t)Q(t)^T \Leftrightarrow T = -\pi I + \alpha_0(I_D)D + \alpha_1(I_D)D^2$
by the First Representation Theorem for Isotropic Tensor Functions.

Gurtin 24.1) Consider the flow of a Reiner-Rivlin fluid between two flat plates. Show that the linear velocity profile remains a solution of the underlying equations with π_0 constant. Show further that, in contrast to the linear theory, the normal stresses are no longer equal.

A Reiner-Rivlin fluid is then incompressible, e.g. $div \quad v = 0$, as well as has constitutive equation $T = -\pi I + \alpha_0(I_D)D + \alpha_1(I_D)D^2$.

Now I use the equation of motion
$\rho_0[v' + (grad \quad v)v] = div \quad T + b$.

I make a side note that:

$$div \quad D = \frac{1}{2} div \quad (grad \quad v + grad \quad v^T) = \frac{1}{2}\Delta v + grad \quad div \quad v = \frac{1}{2}\Delta v$$
$$div \quad D^2 = \partial_i [D_{ik} D_{kj}] = [\partial_i D_{ik}] D_{kj} + D_{ik} [\partial_i D_{kj}] = [div \quad D] D + D \cdot [grad \quad D]$$

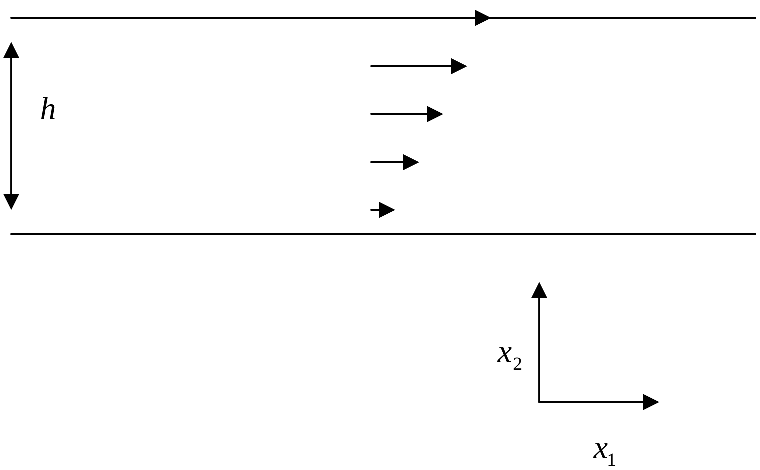
First, I must find

$$div \;\; T = div[-\pi I + \alpha_0(I_D)D + \alpha_1(I_D)D^2]$$
$$= -grad \;\; \pi + [D](grad \;\; \alpha_0(I_D)) + \frac{\alpha_0(I_D)}{2}\Delta v + [D^2](grad \;\; \alpha_1(I_D)) + \alpha_1(I_D)[div \;\; D]D + \alpha_1(I_D)D \cdot [grad \;\; D]$$

Now I consider the linear velocity profile solution, and show that it remains a solution to these equations. This solution is:

$$[grad \quad v] = \begin{bmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with a sample picture



Boundary conditions are a function of plate velocity. Thus, I see that in a linear profile, $\frac{\partial v_1}{\partial x_2} = const$.

Simplifying $div \quad T$ under this solution, then, I get:

$$D = \frac{1}{2}\begin{bmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad and \quad D^2 = \frac{1}{4}\begin{bmatrix} \left(\frac{\partial v_1}{\partial x_2}\right)^2 & 0 & 0 \\ 0 & \left(\frac{\partial v_1}{\partial x_2}\right)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Further, the principal invariants which are a function of $tr \quad D$, $tr \quad D^2$ and $\det \quad D$ become constant. Thus, $\alpha_i(I_D) \rightarrow const$

$$div \quad T =$$
$$= -grad \quad \pi + [D](grad \quad \alpha_0(I_D)) + \frac{\alpha_0(I_D)}{2}\Delta v + [D^2](grad \quad \alpha_1(I_D)) + \alpha_1(I_D)[div \quad D]D + \alpha_1(I_D)D \cdot [grad \quad D]$$
$$\rightarrow -grad \quad \pi$$

However, taking π_0 constant, this simplifies to
$div \quad T = 0$

and the equation of motion becomes, in the absence of body forces,

$$\rho_0[v'+(grad \quad v)v] = div \quad T + b$$
$$\rho_0(grad \quad v)v = 0$$
$$\rho_0\begin{bmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} = 0$$
$$0 = 0$$

So that the linear velocity profile indeed remains a solution.

Now with $T = -\pi I + \alpha_0(I_D)D + \alpha_1(I_D)D^2$, I see that

$$[T] = \begin{bmatrix} -\pi + \frac{\alpha_1}{4}\left(\frac{\partial v_1}{\partial x_2}\right)^2 & \frac{\alpha_0}{2}\left(\frac{\partial v_1}{\partial x_2}\right) & 0 \\ \frac{\alpha_0}{2}\left(\frac{\partial v_1}{\partial x_2}\right) & -\pi + \frac{\alpha_1}{4}\left(\frac{\partial v_1}{\partial x_2}\right)^2 & 0 \\ 0 & 0 & -\pi \end{bmatrix}.$$

Thus, I see that the normal stresses along the diagonal are no longer equal.

Gurtin 25.2) Show that the response functions $\overset{*}{T}$ and $\tilde{T}$ are invariant under $\underline{G}$.

Consider $\underline{G} \subset Orth$. A set $\underline{A} \subset Lin$ is invariant under $\underline{G}$ if $QAQ^T \in \underline{A}$ whenever $A \in \underline{A}$ and $Q \in \underline{G}$. Further, for the symmetry group with $Q \in \underline{G}$, $\hat{T}(FQ, p) = \hat{T}(F, p)$.

$\hat{T}(F)$ is also, according to Gurtin, assumed independent of the observer so that $QTQ^T = \hat{T}(QF)$ is also a member of the constitutive class of the material.

Choosing $Q \in \underline{G}$, then $Q^T \in \underline{G}$ also.

First, I need
$$\hat{T}(F) = \hat{T}(RU) = R\hat{T}(U)R^T = FU^{-1}\hat{T}(U)U^{-1}F^T$$

Next, I can show that $\hat{T}$ is invariant under $\underline{G}$ simply by writing
$$\hat{T}(QFQ^T) = Q\hat{T}(FQ^T)Q^T$$

But, by definition of the symmetry group, $\hat{T}(FQ, p) = \hat{T}(F, p)$ for members of the group so that
$\hat{T}(QFQ^T) = Q\hat{T}(FQ^T)Q^T = Q\hat{T}(F)Q^T$.

I will use this fact to show the remaining identities: As I have shown above, I may take
$\overset{*}{T}(U) = U^{-1}\hat{T}(U)U^{-1}$

Now I may write, for $Q \in Orth, U \in Sym$:
$\overset{*}{T}(QUQ^T) = (QUQ^T)^{-1}\hat{T}(QUQ^T)(QUQ^T)^{-1} = QU^{-1}Q^T Q\hat{T}(U)Q^T QU^{-1}Q^T$
$= QU^{-1}\hat{T}(U)U^{-1}Q^T = Q\overset{*}{T}(U)Q^T$

Next, I will show that $\tilde{T}$ is invariant under $\underline{G}$. Certainly, I may write:
$C = U^2$ so that

$\tilde{T}(QCQ^T) = \tilde{T}(QU^2Q^T) = \tilde{T}((QUQ^T)^2)$

Using the identity $\tilde{T}(C) = \hat{T}\left(C^{\frac{1}{2}}\right)$, wich is easily shown from the definition
$R\tilde{T}(C)R^T = \hat{T}(F) = R\hat{T}(U)R^T$
$\tilde{T}(C) = \hat{T}(U)$
$\tilde{T}(U^2) = \hat{T}(U)$

I get the desired identity:
$\tilde{T}(QCQ^T) = \tilde{T}((QUQ^T)^2) = \hat{T}(QUQ^T) = Q\hat{T}(U)Q^T = Q\tilde{T}(C)Q^T$

Gurtin 27.2) Show that:

$$\hat{S}(FQ) = \hat{S}(F)Q$$

for every Q in the symmetry group $\underline{G}$ and every $F \in Lin^+$. Show further that $\hat{S}$ and $\bar{S}$ are invariant under $\underline{G}$.

We have defined:
$\hat{S}(F) = (\det \quad F)\hat{T}(F)F^{-T}$

so that since by definition for every Q in the symmetry group $\underline{G}$ we have
$\hat{T}(FQ, p) = \hat{T}(F, p)$, I may write:

$$\hat{S}(FQ) = (\det \quad FQ)\hat{T}(FQ)(FQ)^{-T}$$
$$with:$$
$$(\det \quad FQ) = (\det \quad F)(\det \quad Q) = \det \quad F$$
$$\hat{T}(FQ) = \hat{T}(F)$$
$$(FQ)^{-T} = \left(Q^T F^{-1}\right)^T = F^{-T} Q$$
$$and \quad so:$$
$$\hat{S}(FQ) = (\det \quad F)\hat{T}(F)F^{-T}Q = \hat{S}(F)Q$$

Now, using the fact that $\hat{T}\left(QFQ^T\right) = Q\hat{T}(F)Q^T$ as shown above in problem 25.2,

$$\hat{S}(QF) = (\det \quad QF)\hat{T}(QF)(QF)^{-T}$$
$$with:$$
$$(\det \quad QF) = (\det \quad Q)(\det \quad F) = \det \quad F$$
$$\hat{T}(QF) = Q\hat{T}(F)Q^T$$
$$(QF)^{-T} = \left(F^{-1}Q^T\right)^T = QF^{-T}$$
$$and \quad so:$$
$$\hat{S}(QF) = (\det \quad F)Q\hat{T}(F)Q^T QF^{-T} = Q(\det \quad F)\hat{T}(F)F = Q\hat{S}(F)$$

Then, combining the identities $\hat{S}\left(FQ^T\right) = \hat{S}(F)Q^T$ and $\hat{S}(QF) = Q\hat{S}(F)$ as shown above, I have $\hat{S}\left(QFQ^T\right) = Q\hat{S}(F)Q^T$ for all $F \in Lin^+$ and all $Q \in \underline{G}$, so that $\hat{S}$ is invariant under $\underline{G}$.

Next, with $\overline{S}(C) = \sqrt{\det \quad C}\overline{T}(C)$ by definition, I will first show that $\overline{T}(C)$ is invariant under $\underline{G}$ via the procedure that Gurtin used:

With

$$\overline{T}(C) = F^{-1}\hat{T}(F)F^{-T} = U^{-1}R^T\hat{T}(RU)RU^{-1} = U^{-1}R^T R\hat{T}(U)R^T RU^{-1} = U^{-1}\hat{T}(U)U^{-1},$$

I can use the fact that:

$$\left(QUQ^T\right)^2 = QU^2Q^T = QCQ^T$$

To write:

$$Q\overline{T}(C)Q^T = QU^{-1}Q^T Q\hat{T}(U)Q^T QU^{-1}Q^T = \left(QUQ^T\right)^{-1}\hat{T}\left(QUQ^T\right)\left(QUQ^T\right)^{-1}$$
$$= \overline{T}\left(\left(QUQ^T\right)^2\right) = \overline{T}\left(QCQ^T\right)$$

Now, since I have that $\overline{T}(C)$ is invariant under $\underline{G}$, I can write:

$$\bar{S}\left(QCQ^T\right) = \sqrt{\det \quad QCQ^T}\,\bar{T}\left(QCQ^T\right)$$
$$= \sqrt{(\det \quad Q)(\det \quad C)\left(\det \quad Q^T\right)}Q\bar{T}(C)Q^T$$
$$= Q\sqrt{(\det \quad C)}\bar{T}(C)Q^T$$
$$= Q\bar{S}(C)Q^T$$

so I see that $\bar{S}(C)$ is invariant under $\underline{G}$.

Gurtin 27.2) Show that the constitutive equation $S = F\bar{S}(C)$ is independent of observer.

Under a change of observer, $F^* = QF$ and $S^* = QS$. Then, with

$$S = F\bar{S}\left(FF^T\right)$$
$$S = F\sqrt{\det \quad C}\bar{T}(C) = F\sqrt{\det \quad F^T F}\bar{T}\left(F^T F\right)$$

I may then write:

$$S^* = QS = QF\sqrt{\det \quad C}\bar{T}(C) = F^*\sqrt{\det \quad F^T F}\bar{T}\left(F^T F\right)$$
$$But \quad F^T F = F^T Q^T QF = F^{*T} F^*, so:$$
$$S^* = F^*\sqrt{\det \quad F^T F}\bar{T}\left(F^T F\right)$$
$$= F^*\sqrt{\det \quad F^{*T} F^*}\bar{T}\left(F^{*T} F^*\right)$$
$$= F^*\sqrt{\det \quad C^*}\bar{T}\left(C^*\right)$$
$$= F^*\bar{S}\left(C^*\right)$$

so I see that the constitutive class is the same under change of observer.