Ben Sauerwine
Continuum Mechanics Homework 4

In the following exercises, the body B is bounded.

Gurtin 13.2) Other types of momenta of interest are the angular momentum $a_z(t)$ relative to a moving point $z(t)$ and the spin angular momentum $a_{spin}(t)$:

$$a_z(t) = \int_{B_t} [r_z \times v]\rho dV$$
$$a_{spin}(t) = \int_{B_t} [r_\alpha \times v_\alpha]\rho dV$$

Here,

$$r_z(x,t) = x - z(t)$$

is the position vector from $z(t)$, $r_\alpha(x,t)$ is the position vector from $\alpha(t)$, and

$$v_\alpha = \dot{r}_\alpha = v - \dot{\alpha}$$

is the velocity relative to α. For convenience, let $l(t) = l(B,t)$ and $\alpha(t) = \alpha(B,t)$. Show that

$$a_z = a_{spin} + (\alpha - z) \times l$$
$$\dot{a} = \dot{a}_{spin} + (\alpha - o) \times \dot{l}$$

The term $(\alpha - z) \times l$ is usually referred to as the orbital angular momentum about z; it represents the angular momentum B would have if its mass were concentrated at the center of mass.

Recall that the linear momentum and center of mass, respectively, are given by:

$$l(t) = l(B,t) = \int_{B_t} v\rho dV$$

$$\alpha(t) - o = \frac{1}{m(B)} \int_{B_t} r\rho dV$$

First, then, I may obtain:

$$a_z(t) = \int_{B_t} [r_z \times v]\rho dV$$

$$= \int_{B_t} [(r - z(t)) \times v]\rho dV_x$$

$$= \int_{B_t} [(r - \alpha(t) + \alpha(t) - z(t)) \times v]\rho dV_x$$

$$= \int_{B_t} [(r - \alpha(t)) \times v]\rho dV_x + \int_{B_t} [(\alpha(t) - z(t)) \times v]\rho dV_x$$

$$= \int_{B_t} [(r-\alpha(t))\times(v-\dot{\alpha}(t)+\dot{\alpha}(t))]\rho dV_x + (\alpha(t)-z(t))\times\int_{B_t} v\rho dV_x$$
$$= \int_{B_t} [(r-\alpha(t))\times(v-\dot{\alpha}(t))]\rho dV_x + \int_{B_t} [(r-\alpha(t))\times\dot{\alpha}(t)]\rho dV_x + (\alpha(t)-z(t))\times l$$
$$= a_{spin} + \left[\int_{B_t} r\rho dV_x - \alpha(t)\right]\times\dot{\alpha}(t) + (\alpha(t)-z(t))\times l$$
$$= a_{spin} + [\alpha(t)-\alpha(t)]\times\dot{\alpha}(t) + (\alpha(t)-z(t))\times l$$
$$= a_{spin} + (\alpha(t)-z(t))\times l$$

Now, selecting $z(t)=o$ as the origin, I may write:

$$a = a_{spin} + (\alpha - o)\times l$$
$$differentiating,$$
$$\dot{a} = \dot{a}_{spin} + (\alpha-o)^{\bullet}\times l + (\alpha-o)\times\dot{l}$$
$$= \dot{a}_{spin} + \dot{\alpha}\times l + (\alpha-o)\times\dot{l}$$
$$= \dot{a}_{spin} + l\times l + (\alpha-o)\times\dot{l}$$
$$= \dot{a}_{spin} + (\alpha-o)\times\dot{l}$$

Gurtin 13.3) Consider a rigid motion. Then,

$$x_0(y,t) = q(t) + Q(t)(y-z)$$

for all $y \in B_0$ and all times t.

(a) Show that:

$$\alpha(t) = q(t) + Q(t)[a(0)-z]$$

and hence, noting that $x_0(y,t)$ is well-defined for all $y \in E$,

$$\alpha(t) = x_0(\alpha(0),t)$$

What is the meaning of this result?

$$\alpha(t)-o = \frac{1}{m(B)}\int_{B_t}(x_0(y,t)-o)\rho dV_x = \frac{1}{m(B)}\int_{B_0}[q(t)-o+Q(t)(y-z)]\rho dV_y$$
$$= (q(t)-o)\left[\frac{1}{m(B)}\int_{B_0}\rho dV_y\right] - Q(t)z\left[\frac{1}{m(B)}\int_{B_0}\rho dV_y\right] + Q(t)\left[\frac{1}{m(B)}\int_{B_0} y\rho dV_y\right]$$
$$= (q(t)-o) - Q(t)z + Q(t)\left[\frac{1}{m(B)}\int_{B_0} y\rho dV_y\right]$$

However, since y is a continuous spatial field, I have:

$$\int_{P_t} \Phi(x,t)\rho(x,t)dV_x = \int_{P_t} \Phi_m(p,t)\rho_0(p)dV_p$$
$$\frac{1}{m(B)}\int_{B_t} y\rho dV_y = \frac{1}{m(B)}\int_{B_0} y\rho dV_y = \alpha(0) - o$$

and so

$$\alpha(t) - o = q(t) - Q(t)z + Q(t)[\alpha(0) - o]$$
$$(\alpha(t) - o) = q(t) + Q(t)[(\alpha(0) - o) - z]$$
$$\alpha(t) = x_0(\alpha(0), t)$$

Thus, I see that the motion of the center of mass of a rigid object can be treated independently of its components, as is done so often in elementary mechanics. Further, the same material point occupies the location $\alpha(t)$ for all t.

(b) Show that the angular velocity $\omega(t)$ is the axial vector of $\dot{Q}(t)Q(t)^T$ and that $v_\alpha = \omega \times r_\alpha$.

From part a above and the given that $x_0(y,t) = q(t) + Q(t)(y - z)$, then I have:

$$x_0(y,t) - \alpha(t) = Q(t)(y - \alpha(0))$$
$$Q(t)^T[x_0(y,t) - \alpha(t)] = y - \alpha(0)$$

But, using the fact (shown in assignment 3, problem 9.1a) that $v(x_0(y,t),t) = \dot{x}_0(y,t)$, then:

$$v(x_0(y,t),t) = \dot{x}_0(y,t) = \dot{\alpha}(t) + \dot{Q}(t)[y - \alpha(0)]$$

Substituting,

$$\dot{x}_0(y,t) = \dot{\alpha}(t) + \dot{Q}(t)Q(t)^T[x_0(y,t) - \alpha(t)]$$
$$\dot{x}_0(y,t) - \dot{\alpha}(t) = \dot{Q}(t)Q(t)^T[x_0(y,t) - \alpha(t)]$$
$$v_\alpha = \dot{Q}(t)Q(t)^T r_\alpha$$

As seen in problem 3.6 from an earlier assignment, however, $Q(t)\dot{Q}(t)^T$ is skew and so $\left(Q(t)\dot{Q}(t)^T\right)^T = \dot{Q}(t)Q(t)^T$ is skew as well. However, there is a one-to-one correspondence between skew tensors and vectors, so this uniquely defines a $\omega(t)$, the axial vector of $Q(t)\dot{Q}(t)^T$, such that $v_\alpha = \omega \times r_\alpha$.

(c) A vector function k on R rotates with the body if

$$k(t) = Q(t)k(0)$$

for all t. Note that $Q(0) = I$ since $x_0(y,0) = y$ for all y. Show that k rotates with the body if and only if

$$\dot{k}(t) = \omega \times k$$

If $\dot{k}(t) = \omega(t) \times k(t)$, then from part (b) $\dot{k}(t) = \dot{Q}(t)Q(t)^T k(t)$. Further, from the condition that $Q(0) = I$, I have that $\dot{k}(0) = \dot{Q}(0)k(0)$. These equations form a set of ordinary differential equations which have a unique solution. Testing $k(t) = Q(t)k(0)$ as a solution, I certainly have $\dot{k}(t) = \dot{Q}(t)k(0)$ and so

$$\begin{aligned}
&\dot{k}(t) = \omega(t) \times k(t) \\
&\dot{Q}(t)k(0) = \dot{Q}(t)Q(t)^T k(t) \\
&\dot{Q}(t)k(0) = \dot{Q}(t)Q(t)^T Q(t)k(0) \\
&\dot{Q}(t) = \dot{Q}(t)
\end{aligned}$$

Thus, working in the if direction, I see that $k(t) = Q(t)k(0)$ is a valid solution to the ordinary differential equation set $\dot{k}(t) = \omega \times k$, $\dot{k}(0) = \dot{Q}(0)k(0)$ this hypothesis then holds in the if direction.

Working in the only if direction, I notice that $k(t) = Q(t)k(0)$ is the unique solution to the set of ordinary differential equations $\dot{k}(t) = \omega \times k$ and $\dot{k}(0) = \dot{Q}(0)k(0)$ as seen above, and therefore these differential equations hold for any system where k rotates with the body.

(d) Use the identity

$$f \times (d \times f) = \left(f^2 I - f \otimes f\right)d$$

to show that

$$a_{spin} = J\omega$$

where

$$J(t) = \int_{B_t} \left(r_\alpha^2 I - r_\alpha \otimes r_\alpha\right)\rho dV$$

is the inertia tensor of B_t relative to the center of mass.

Using the result from problem 13.2 above, I have that:

$$a_{spin}(t) = \int_{B_t} [r_\alpha \times v_\alpha]\rho dV$$

$$= \int_{B_t} [r_\alpha \times (\omega \times r_\alpha)]\rho dV$$

$$= \int_{B_t} [r_\alpha^2 I - (r_\alpha \times r_\alpha)]\omega\rho dV = \left[\int_{B_t} [r_\alpha^2 I - (r_\alpha \times r_\alpha)]\rho dV\right]\omega = J\omega$$

(e) Show that

$$J(t) = Q(t)J(0)Q(t)^T$$

and use this fact to prove that the matrix $[J(t)]$ of $J(t)$ relative to any orthonormal basis $\{e_i(t)\}$ that rotates with the body is independent of t.

Note that in the definition

$$J(t) = \int_{B_t} (r_\alpha^2 I - r_\alpha \otimes r_\alpha)\rho dV \ ,$$

$r_\alpha^2 I - r_\alpha \otimes r_\alpha$ is a continuous spatial field.

Then,

$$J(t) = \int_{B_t} (r_\alpha^2 I - r_\alpha \otimes r_\alpha)\rho dV = \int_{B_0} (r_\alpha^2 I - r_\alpha \otimes r_\alpha)\rho dV \ .$$

But, as the body must inherently rotate with itself, I have $r_\alpha(x_0(y,t),t) = Q(t)r_\alpha(y,0)$.

Then,

$$J(t) = \int_{B_t} (r_\alpha^2(t)I - r_\alpha(t) \otimes r_\alpha(t))\rho dV = \int_{B_0} (r_\alpha(t) \cdot r_\alpha^T(t)I - r_\alpha(t) \otimes r_\alpha(t))\rho dV$$

$$= \int_{B_0} Q(t)(r_\alpha^2(0)I - r_\alpha(0) \otimes r_\alpha(0))Q^T(t)\rho dV = Q(t)\left[\int_{B_0} (r_\alpha^2(0)I - r_\alpha(0) \otimes r_\alpha(0))\rho dV\right]Q^T(t)$$

$$= Q(t)J(0)Q^T(t)$$

However, in the rotating orthonormal basis $\{e_i(t)\}$, I have shown in part c that

$$e(t) = Q(t)e(0)$$

so that:

$$e_i(t) \cdot J(t)e_j(t) = e_i(0) \cdot [Q(t)^T Q(t)J(0)Q(t)^T Q(t)]e_j(0) = e_i(0) \cdot [J(0)]e_j(0)$$

Thus I see that in the rotating basis, the matrix $e_i(t) \cdot J(t)e_j(t)$ collapses to $e_i(0) \cdot J(0)e_j(0)$, which is then independent of time.

(f) Construct an orthonormal basis $\{e_i(t)\}$ that rotates with the body and has

$$[J] = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$$

In this case $\{e_i(t)\}$ is called a principal basis and the corresponding numbers J_i are called moments of inertia. Let $\omega_i(t)$ denote the components of $\omega(t)$ with respect to $\{e_i(t)\}$. Show that the components of $\dot{a}_{spin}(t)$ relative to this basis are given by

$$\begin{aligned}(\dot{a}_{spin})_1 &= J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_3 \\ (\dot{a}_{spin})_2 &= J_2\dot{\omega}_2 + (J_1 - J_3)\omega_1\omega_3 \\ (\dot{a}_{spin})_3 &= J_3\dot{\omega}_3 + (J_2 - J_1)\omega_1\omega_2\end{aligned}$$

It is clear that such a basis must exist since I may choose a vector $\bigcup_{i=1}^{3}\{e_i(0)\}$ consisting of eigenvectors of the symmetric tensor $J(0)$. By part e, with $e_i(t) = Q(t)e_i(0)$, then $[J(t)]_{\{e_1,e_2,e_e\}|_t}$ is independent of time and diagonal.

With the help of part d, I have

$$a_{spin}(t) = J\omega = \sum_{i,j} J_{ij}\omega_j(t)e_i(t)$$

And from part c, I have (ε_{ikl} is the Levi-Civita symbol)

$$\dot{a}_{spin}(t) = J\omega = \sum_{i,j} J_{ij}\dot{\omega}_j(t)e_i(t) + J_{ij}\omega_j(t)(\omega \times e(t))_i = \sum_{i,j} J_{ij}\dot{\omega}_j(t)e_i(t) + J_{ij}\omega_j(t)\varepsilon_{ikl}\omega_k(t)e_l(t)$$

However, in the orthonormal basis that rotates with the body described above, the basis $\{e_i(t)\}$ diagonalizes $[J]$, giving:

$$\dot{a}_{spin}(t) = \sum_{i} J_i\dot{\omega}_i(t)e_i(t) + J_i\omega_i(t)\varepsilon_{ijk}\omega_j(t)e_k(t)$$

This is component-wise the same as the result above.

Gurtin 14.1) The moment $m_z(t)$ about a moving point $z(t)$ is defined by $m_z(t) = \int_{\partial B_t} r_z \times s(n)dA + \int_{B_t} r_z \times b dV$ where r_z is the position vector from z.

(a) Let $f(t) = f(B,t)$. Show that for $y : R \to E$,

$$m_z = m_y + (y - z) \times f$$

By definition,

$$r_z(x,t) = x - z(t) = [x - y(t)] + [y(t) - z(t)]$$

Under this substitution, then,

$$m_z(t) = m_y(t) + \int_{\partial B_t} [y(t) - z(t)] \times s(n) dA + \int_{B_t} [y(t) - z(t)] \times b dV$$
$$= m_y(t) + [y(t) - z(t)] \times \left[\int_{\partial B_t} s(n) dA + \int_{B_t} b dV \right]$$
$$= m_y(t) + [y(t) - z(t)] \times f(B,t)$$

(b) Let $l(t) = l(B,t)$**. Show that**

$$m_z = \dot{a}_z + \dot{z} \times l$$
$$m_\alpha = \dot{a}_{spin}$$

By definition,

$$r_z(x,t) = x - z(t)$$
$$\dot{r}_z(x,t) = v - \dot{z}(t)$$

Then,

$$\dot{a}_z = \left(\int_{B_t} r_z \times v \rho dV \right)^{\bullet}$$
$$= \int_{B_t} r_z \times \dot{v} \rho dV + \int_{B_t} (v - \dot{z}(t)) \times v \rho dV$$
$$= \int_{B_t} r_z \times \dot{v} \rho dV - \int_{B_t} \dot{z}(t) \times v \rho dV$$
$$= \int_{B_t} r_z \times \dot{v} \rho dV - \dot{z}(t) \times \int_{B_t} v \rho dV$$
$$= \left[\int_{B_t} r_z \times \dot{v} \rho dV \right] - \dot{z}(t) \times l$$
$$= \left[\int_{\partial B_t} r_z \times s(n) dA + \int_{B_t} r_z \times b dV \right] - \dot{z}(t) \times l$$
$$= m_z - \dot{z}(t) \times l$$

Then,

$\dot{a}_z + \dot{z}(t)\times l = m_z$.

However, if $z(t) = \alpha(t)$, then

$$m_\alpha = \dot{a}_\alpha + \dot{\alpha}(t)\times l = \dot{a}_\alpha + \frac{1}{m(B)} l \times l = \dot{a}_\alpha = \dot{a}_{spin}$$

In the final step I have used the result of problem 13.2 above.

Gurtin 14.4) Consider a rigid motion with angular velocity ω. Let $\{e_i(t)\}$ denote a principal basis of inertia and let J_i denote the corresponding moments of inertia (relative to α). Further, let $\omega_i(t)$ and $m_i(t)$ denote the components of $\omega(t)$ and $m_\alpha(t)$ relative to $\{e_i(t)\}$. Derive Euler's equations,

$$\begin{aligned} m_1 &= J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_3 \\ m_2 &= J_2\dot{\omega}_2 + (J_1 - J_3)\omega_1\omega_3 \\ m_3 &= J_3\dot{\omega}_3 + (J_2 - J_1)\omega_1\omega_2 \end{aligned}$$

These equations supplemented by

$$f = m(B)\ddot{\alpha}$$

constitute the basic equations of rigid body mechanics. When f and m are known they provide a system of nonlinear ordinary differential equations for ω and α.

This is a relatively straightforward proof, using things I have already proven.

From the result of 14.1b above, I have that $m_\alpha = \dot{a}_{spin}$.

From the result of 13.3f above, I have that $\dot{a}_{spin}(t) = \sum_i J_i\dot{\omega}_i(t)e_i(t) + J_i\omega_i(t)\varepsilon_{ijk}\omega_j(t)e_k(t)$.

Then, $m_\alpha = \dot{a}_{spin} = \sum_i J_i\dot{\omega}_i(t)e_i(t) + J_i\omega_i(t)\varepsilon_{ijk}\omega_j(t)e_k(t)$.

Gurtin 14.5) Let T be the stress at a particular place and time. Suppose that the corresponding surface force on a given plane S is perpendicular to S, while the surface force on any plane perpendicular to S vanishes. Show that T is a pure tension.

Let e_1 be the surface normal to S , and e_2 and e_3 comprise the rest of an orthonormal basis $\{e_i\}$.

Then, since the surface force on S is perpendicular to S , then $s(e_1) = Te_1 = \sigma e_1$.

Selecting the planes perpendicular to S in the directions of e_2 and e_3, then
$s(e_2) = s(e_3) = Te_2 = Te_3 = 0$.

Now, the equations
$Te_1 = \sigma e_1 \quad Te_2 = 0 \quad Te_3 = 0$
constrain that $T = \sigma e_1 \otimes e_1$, which is by definition a pure tension.

Gurtin 14.9) Suppose that at time t the surface traction vanishes on the boundary ∂B_t. Show that at any $x \in \partial B_t$ the stress vector on any plane perpendicular to ∂B_t is tangent to the boundary.

On the boundary with surface normal n, then, $Tn = 0$. Then consider some unit vector k tangent to the boundary and thus orthogonal to n:

$$Tn = 0$$
$$Tn \cdot k = 0$$
$$n \cdot T^T k = 0$$

But, since T is symmetric, $T = T^T$ and
$n \cdot Tk = 0$

Thus, since the vector k was arbitrary and orthogonal to n, I see that no plane perpendicular to the boundary has any component of stress in the direction of the boundary n.

Gurtin 15.1) Consider a statical situation in which a (bounded) body occupies the region B_0 for all time. Let $b : B_0 \to V$ and $T : B_0 \to Sym$ with T smooth satisfy

$$div \quad T + b = 0$$

Define the mean stress $\overline{T}$ through

$$vol(B_0)\overline{T} = \int_{B_0} T dV$$

(a) **(Signorini's Theorem) Show that $\overline{T}$ is completely determined by the surface traction Tn and the body force b as follows:**

$$vol(B_0)\overline{T} = \int_{\partial B_0} (Tn \otimes r) dA + \int_{B_0} (b \otimes r) dV$$

From problem 5.1b, from assignment #2, I have:
$$\int_{\partial R} (Sn) \otimes v dA = \int_R \left[(div \quad S) \otimes v + S \nabla v^T \right] dV$$

Let:

$v \to r, \quad S \to T, \quad R \to B_0$

Then:

$$\int_{\partial B_0} (Tn) \otimes r dA = \int_{B_0} \left[(div \quad T) \otimes r + T \nabla r^T \right] dV$$

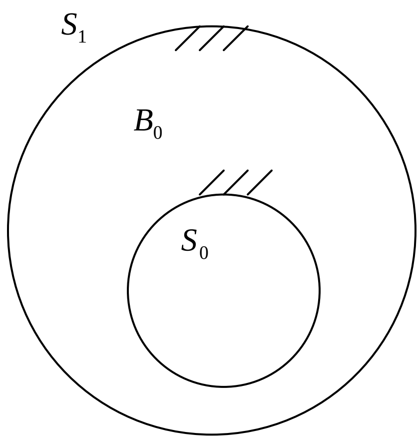
$$\int_{\partial B_0} (Tn) \otimes r dA = \int_{B_0} \left[-b \otimes r + T \right] dV$$

$$\int_{\partial B_0} (Tn) \otimes r dA + \int_{B_0} b \otimes r dV = \int_{B_0} T dV = vol(B_0) \overline{T}$$

as expected.

(b) Assume that $b = 0$ and that ∂B_0 consists of two closed surfaces S_0 and S_1 with S_1 enclosing S_0 (Fig. 10). Assume further that S_0 and S_1 are acted on by uniform pressures π_0 and π_1, so that

$$s(n) = -\pi_0 n \quad on \quad S_0$$
$$s(n) = -\pi_1 n \quad on \quad S_1$$



with π_0 and π_1 constants. Show that $\overline{T}$ is a pressure of amount:

$$\frac{\pi_1 v_1 - \pi_0 v_0}{v_1 - v_0}$$

where v_0 and v_1 are, respectively, the volumes enclosed by S_0 and S_1.

Here, let subscripts on the normal n indicate the surface the normal is taken with respect to.

With $b = 0$, I have from part (a):

$$vol(B_0)\overline{T} = \int_{S_0} (Tn_{B_0} \otimes r) dA + \int_{S_1} (Tn_{B_0} \otimes r) dA$$

noting the surface normal orientations,

$$= -\pi_0 \int_{S_0} (n_{S_0} \otimes r) dA - \pi_1 \int_{S_1} (n_{S_1} \otimes r) dA$$

From assignment #2, problem 5.1a, I saw that:

$\int_{\partial R} v \otimes n_R dA = \int_R \nabla v dV \quad (n \quad normal \quad to \quad R)$, or equivalently $\int_{\partial R} n \otimes v dA = \int_R \nabla v^T dV$.

Then, letting $v \to r$ in the equations above,

$$vol(B_0)\overline{T} = \pi_0 \int_{S_0} ((-n) \otimes r) dA - \pi_1 \int_{S_1} (n \otimes r) dA$$

$$= -\pi_0 I \int_{V_0} dV + \pi_1 I \int_{V_1} dV$$

$$= -(\pi_0 v_0 + \pi_1 v_1) I$$

However, looking at the picture above, I see that $vol(B_0) = v_1 - v_0$, so that

$$\overline{T} = \frac{\pi_1 v_1 - \pi_0 v_0}{v_1 - v_0}$$

Gurtin 15.3) Derive the following formula for a control volume R .

$$\int_{\partial R} s(n) \cdot v dA + \int_R b \cdot v dV = \int_R T \cdot D dV + \frac{d}{dt} \int_R \frac{v^2}{2} \rho dV + \int_{dR} \frac{\rho v^2}{2} (v \cdot n) dA$$

Note that $D = \frac{1}{2}\left(grad \quad v + grad \quad v^T\right)$

This is an analog of the theorem of power expended for a material part:

$$\int_{\partial P_t} s(n) \cdot v dA + \int_{P_t} b \cdot v dV = \int_{P_t} T \cdot D dV + \frac{d}{dt} \int_{P_t} \frac{v^2}{2} \rho dV$$

The extra term merely accounts for the kinetic energy of matter entering and leaving the control volume.

Since T is symmetric due to momentum balance, then since D is symmetric,

$T \cdot grad \quad v = T \cdot D$

Next, I will use the theorem of conservation of mass for a control volume to write:

$$\frac{d}{dt}\int_R \frac{v^2}{2}\rho dV$$

$$= \int_R \left[\frac{d}{dt}\frac{v^2}{2}\right]\rho dV + \int_R \frac{v^2}{2}\left[\frac{d}{dt}\rho dV\right]$$

$$= \int_R \left[\frac{d}{dt}\frac{v^2}{2}\right]\rho dV - \int_{\partial R} \frac{v^2}{2}\rho(v \cdot n)dA$$

$$= \int_R v \cdot \dot{v}\rho dV - \int_{\partial R} \frac{v^2}{2}\rho(v \cdot n)dA$$

Using the definition of total body force $b_* = b - \rho\dot{v}$, then, I may write

$$\frac{d}{dt}\int_R \frac{v^2}{2}\rho dV$$

$$= \int_R v \cdot \dot{v}\rho dV - \int_{\partial R} \frac{v^2}{2}\rho(v \cdot n)dA$$

$$= -\int_R b_* \cdot v + \int_R b \cdot v - \int_{\partial R} \frac{v^2}{2}\rho(v \cdot n)dA$$

So that:

$$\int_R b_* \cdot v = -\frac{d}{dt}\int_R \frac{v^2}{2}\rho dV + \int_R b \cdot v - \int_{\partial R} \frac{v^2}{2}\rho(v \cdot n)dA$$

However,

$$\int_{\partial R} Tn \cdot v dA$$

$$= \int_{\partial R} T^T v \cdot n dA$$

$$= \int_R div\left(T^T v\right)dV$$

$$= \int_R \left[v \cdot (div \quad T) + T \cdot (grad \quad v)\right]dV$$

Then, under the substitutions $Tn = s(n)$ and $div \quad T = -b_*$. Thus, I have:

$$\int_{\partial R} s(n) \cdot v dA = -\int_R b_* \cdot v dV + \int_R T \cdot (grad \quad v)dV$$

Substituting, then, I have:

$$\int_{\partial R} s(n)\cdot v dA = -\int_R b_* \cdot v dV + \int_R T\cdot(grad \quad v)dV$$

$$\int_{\partial R} s(n)\cdot v dA = \frac{d}{dt}\int_R \frac{v^2}{2}\rho dV - \int_R b\cdot v + \int_{\partial R}\frac{v^2}{2}\rho(v\cdot n)dA + \int_R T\cdot(grad \quad v)dV$$

$$\int_{\partial R} s(n)\cdot v dA = \frac{d}{dt}\int_R \frac{v^2}{2}\rho dV - \int_R b\cdot v + \int_{\partial R}\frac{v^2}{2}\rho(v\cdot n)dA + \int_R T\cdot D dV$$

$$\int_{\partial R} s(n)\cdot v dA + \int_R b\cdot v = \frac{d}{dt}\int_R \frac{v^2}{2}\rho dV + \int_{\partial R}\frac{v^2}{2}\rho(v\cdot n)dA + \int_R T\cdot D dV$$

Let a motion χ of B, w_0 and ω in V, $t\in R$ and be given, and define as in class:

$$w(x) = w_0 + \omega\times r(x) = w_0 + \omega\times(x-o)$$
$$\chi_\omega(p,\tau) = o + (\tau - t)w_0 + Q_\omega(\tau)(\chi(p,\tau)-o)$$

where $\tau \mapsto Q_\omega(\tau)\in Orth^+$ is the unique solution of the initial value problem:

$$\dot{Q}_\omega(\tau) = WQ_\omega(\tau) \text{ **for all** } \tau\in R \text{ **and** } Q_\omega(t) = I$$

Here, W is the skew tensor whose axial vector is the given vector ω, so that $Wa = \omega\times a$ for all a in V. Verify the following relations:

$$\chi_\omega(p,t) = \chi(p,t) \text{ **for all** } p\in B$$

and, for all $x\in\chi(B,t)$,

$$\rho_\omega(x,t) = \rho(x,t)$$
$$v_\omega(x,t) = v(x,t) + w(x)$$
$$\dot{v}_\omega(x,t) = \dot{v}(x,t) + 2\omega\times v(x,t) + \omega\times(\omega\times r(x))$$

Starting with
$\chi_\omega(p,\tau) = o + (\tau - t)w_0 + Q_\omega(\tau)(\chi(p,\tau)-o)$,

I evaluate at $\tau = t$:
$\chi_\omega(p,t) = o + (t-t)w_0 + Q_\omega(t)(\chi(p,t)-o)$
$= o + Q_\omega(t)(\chi(p,t)-o)$

However, $Q_\omega(t) = I$ by definition, so that:
$\chi_\omega(p,t) = o + I(\chi(p,t)-o)$
$= \chi(p,t)$

Since $\chi_\omega(p,t) = \chi(p,t)$, then, these functions must refer to the same material point and it must be the case for all $p \in B$ that the density $\rho_\omega(x,t) = \rho(x,t)$.

$$
\begin{aligned}
v(x,t) &= \dot{\chi}_\omega(p,t) = \left[\dot{\chi}_\omega(p,\tau)\right]_{\tau=t} \\
&= \left[o + (\tau - t)w_0 + Q_\omega(\tau)(\chi(p,\tau) - o)\right]^{\bullet}{}_{\tau=t} \\
&= \left[w_0 + \dot{Q}_\omega(\tau)(\chi(p,\tau) - o) + Q_\omega(\tau)\dot{\chi}(p,\tau)\right]_{\tau=t} \\
&= w_0 + \dot{Q}_\omega(t)(\chi(p,t) - o) + Q_\omega(t)\dot{\chi}(p,t) \\
&= w_0 + WQ_\omega(t)(\chi(p,t) - o) + \dot{\chi}(p,t) \\
&= w_0 + \omega \times (\chi(p,t) - o) + v(x,t) \\
&= w(x) + v(x,t)
\end{aligned}
$$

Further,

$$
\begin{aligned}
\dot{v}(x,t) &= \ddot{\chi}_\omega(p,t) = \left[\ddot{\chi}_\omega(p,\tau)\right]_{\tau=t} \\
&= \left[o + (\tau - t)w_0 + Q_\omega(\tau)(\chi(p,\tau) - o)\right]^{\bullet\bullet}{}_{\tau=t} \\
&= \left[w_0 + \dot{Q}_\omega(\tau)(\chi(p,\tau) - o) + Q_\omega(\tau)\dot{\chi}(p,\tau)\right]^{\bullet}{}_{\tau=t} \\
&= \left[\ddot{Q}_\omega(\tau)(\chi(p,\tau) - o) + \dot{Q}_\omega(\tau)\dot{\chi}(p,\tau) + \dot{Q}_\omega(\tau)\dot{\chi}(p,\tau) + Q_\omega(\tau)\ddot{\chi}(p,\tau)\right]_{\tau=t} \\
&= \left[W^2(\chi(p,\tau) - o) + 2W\dot{\chi}(p,\tau) + Q_\omega(\tau)\ddot{\chi}(p,\tau)\right]_{\tau=t} \\
&= \left[\omega \times (\omega \times (\chi(p,\tau) - o)) + 2\omega \times \dot{\chi}(p,\tau) + Q_\omega(\tau)\ddot{\chi}(p,\tau)\right]_{\tau=t} \\
&= \omega \times (\omega \times (\chi(p,t) - o)) + 2\omega \times \dot{\chi}(p,t) + Q_\omega(t)\ddot{\chi}(p,t) \\
&= \omega \times (\omega \times r(x)) + 2\omega \times v(p,t) + \dot{v}(p,t)
\end{aligned}
$$