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Continuum Mechanics Homework 3

**(Gurtin 8.1) A motion is a simple shear if the velocity field has the form**  
 $v(x,t) = v_1(x_2)e_1$  **in some Cartesian frame. Show that for a simple shear,**  
 $div\ v = 0$   $(grad\ v)v = 0$   $\dot{v} = v'$ .

Note that in order for divergence to be nonzero, the  $i$ -th component of  $v$  must have a dependence on the  $i$ -th component of  $x$ .

$$div\ v = tr\ \nabla v = \sum_i \frac{\partial v_i}{\partial x_i} = \frac{\partial}{\partial x_1} [v_1(x_2)] = 0$$

Allowing the tensor from the gradient to act, I have:

$$(grad\ v)v = \left[ \left( \frac{\partial}{\partial x_1} v_1(x_2)e_1 \otimes e_1 + \frac{\partial}{\partial x_2} v_1(x_2)e_1 \otimes e_2 + \frac{\partial}{\partial x_3} v_1(x_2)e_1 \otimes e_3 \right) \right] v_1(x_2)e_1$$

$$= \begin{bmatrix} 0 & v_1'(x_2) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(x_2) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Finally, since  $\dot{v} = v' + (grad\ v)v$  and  $(grad\ v)v = 0$ ,  $\dot{v} = v'$ .

**(Gurtin 8.3) Let  $D$  and  $W$  be, respectively, the symmetric and skew parts of**  
 $grad\ v$ . **Let  $v$  be a class  $C^2$  velocity field. Show that**

$$div\ \dot{v} = (div\ v)^\bullet + |D|^2 - |W|^2.$$

$$div\ \dot{v} = div\ (v' + (grad\ v)v) = div\ (v') + div\ ((grad\ v)v)$$

First I will derive the identity from exercise 4.9 (a),

$$div\ (grad\ v)v = \partial^i \left[ \left[ \partial_i v^j \right] v_j \right] = \left[ \partial_i v^j \right] \left[ \partial^i v_j \right] + v_j \partial^i \left[ \partial_i v^j \right] = \left[ \partial_i v^j \right] \left[ \partial^i v_j \right] + v_j \partial_i \left[ \partial^i v^j \right]$$

$$= (grad\ v) \cdot (grad\ v)^T + v \cdot (grad\ (div\ v))$$

Then,

$$div\ \dot{v} = div\ (v' + (grad\ v)v)$$

$$= div\ v' + (grad\ v) \cdot (grad\ v)^T + v \cdot (grad\ (div\ v))$$

However, in an exercise on a previous assignment (problem Gurtin 7.4) I have shown that

$$(grad\ v) \cdot (grad\ v)^T = |D|^2 - |W|^2. \text{ Then,}$$

$$\operatorname{div} \dot{v} = [\operatorname{div} (v') + v \cdot (\operatorname{grad} (\operatorname{div} v))] + |D|^2 - |W|^2$$

$$\operatorname{div} \dot{v} = [(\operatorname{div} v)'] + |D|^2 - |W|^2$$

**(Gurtin 8.4) Consider the motion of  $E$  defined by:**

$$x_1 = p_1 e^t$$

$$x_2 = p_2 + t$$

$$x_3 = p_3$$

**in some Cartesian frame. Compute the spatial velocity field  $v$  and determine the streamlines.**

$$v_1 = \dot{x}_1 = p_1 e^t = x_1$$

$$v_2 = \dot{x}_2 = 1$$

$$v_3 = \dot{x}_3 = 0$$

Recall that:

$$\dot{s}_i(\lambda) = v_i \text{ under the substitution } \{x_i(t) \rightarrow s_i(\lambda)\}.$$

Then,

$$\dot{s}_1(\lambda) = s_1(\lambda)$$

$$\dot{s}_2(\lambda) = 1$$

$$\dot{s}_3(\lambda) = 0$$

So that clearly the solutions to this passing through  $(y_1, y_2, y_3)$  are:

$$s_1(\lambda) = y_1 e^\lambda$$

$$s_2(\lambda) = y_2 + \lambda$$

$$s_3(\lambda) = y_3$$

**(Gurtin 8.5) Consider the motion  $x$  defined by:**

$$x(p, t) = p_0 + U(t)[p - p_0]$$

**where**

$$U(t) = \sum_{i=1}^3 \alpha_i(t) e_i \otimes e_i$$

**with  $\alpha_i(t) > 0$  smooth. (Here,  $\{e_i\}$  is an orthonormal basis.) Compute  $p$ ,  $v$ , and  $L$  and determine the streamlines.**

Let the orthonormal basis  $\{e_i\}$  be independent of time.

$$\text{First, clearly } x_i(p, t) = p_{0,i} + \alpha_i(t)[p_i - p_{0,i}].$$

Then, solving and using the fact that  $\alpha_i(t) > 0$ ,  $p_i = \frac{1}{\alpha_i(t)}[x_i - p_{0,i}] + p_{0,i}$ .

Next,  $v_i = \dot{x}_i(p, t) = \alpha_i'(t)[p_i - p_{0,i}] = \frac{\alpha_i'(t)}{\alpha_i(t)}[x_i - p_{0,i}]$ , where I have substituted from above and used the fact that  $\alpha_i(t) > 0$  and is smooth.

This makes

$$[L(x, t)] = \begin{bmatrix} \frac{\alpha_1'(t)}{\alpha_1(t)} & 0 & 0 \\ 0 & \frac{\alpha_2'(t)}{\alpha_2(t)} & 0 \\ 0 & 0 & \frac{\alpha_3'(t)}{\alpha_3(t)} \end{bmatrix}$$

Streamlines are again given by:

$$\dot{s}_i(\lambda) = v_i \text{ under the substitution } \{x_i(t) \rightarrow s_i(\lambda)\}.$$

so that

$$\dot{s}_i(\lambda) = \frac{\alpha_i'(t)}{\alpha_i(t)}[s_i(\lambda) - p_{0,i}]$$

$\therefore$

$$s_i(\lambda) = y_i e^{\frac{\alpha_i'(t)}{\alpha_i(t)}\lambda} - \frac{\alpha_i'(t)}{\alpha_i(t)} p_{0,i} \lambda$$

$$s_i(0) = y_i$$

### (Gurtin 9.1)

*For this problem, I use the vector symbol to distinguish between specific points in three-space and the reference maps. The latter will have no vector symbol. Further, define  $\nabla_{\bar{p}}$  to mean "the gradient with respect to vector  $\bar{p}$ ".*

It is often convenient to label material points by their positions at a given time  $\tau$ . Suppose that a material point  $\bar{p}$  occupies the place  $\bar{y}$  at  $\tau$  and  $\bar{x}$  at an arbitrary time  $t$ .

$$\bar{y} = x(\bar{p}, \tau) \quad \bar{x} = x(\bar{p}, t)$$

**Roughly speaking, we want  $\bar{x}$  as a function of  $\bar{y}$ . Thus, since  $\bar{p} = p(\bar{y}, \tau)$  we have  $\bar{x} = x(p(\bar{y}, \tau), t)$ .**

**We call the function  $x_\tau : B_\tau \times R \rightarrow E$  defined by  $x_\tau(\bar{y}, t) = x(p(\bar{y}, \tau), t)$  the motion relative to time  $\tau$ ;  $x_\tau(\bar{y}, t)$  is the place occupied at time  $t$  by the material point that occupies point  $\bar{y}$  at time  $\tau$ . Let  $F_\tau(\bar{y}, t) = \nabla_y x_\tau(\bar{y}, t)$ , where  $\nabla_y$  is the gradient with respect to  $\bar{y}$  holding  $t$  fixed. Also, let  $F_\tau = R_\tau U_\tau$  denote the right polar decomposition of  $F_\tau$ , and define  $C_\tau = (U_\tau)^2$ .**

**a) Show that  $v(\bar{x}, t) = \frac{\partial}{\partial t} x_\tau(\bar{y}, t)$  provided  $\bar{x} = x_\tau(\bar{y}, t)$ .**

Note that:

$$x(\bar{p}, t) = x_\tau(x(\bar{p}, \tau), t)$$

$$\bar{y} = x(\bar{p}, \tau)$$

Then, what this is saying that the velocity at some absolute time should be the same irrespective of one's definition of 'zero' of time.

Then:

$$v(\bar{x}, t) \equiv \frac{\partial}{\partial t} x(\bar{p}, t) = \frac{\partial}{\partial t} x_\tau(x(\bar{p}, \tau), t) = \frac{\partial}{\partial t} x_\tau(\bar{y}, t)$$

**b) Use the relation  $x(\cdot, t) = x_\tau(\cdot, t) \circ x(\cdot, \tau)$  to show that  $F_\tau(\bar{y}, t)F(\bar{p}, \tau) = F(\bar{p}, t)$ , where  $\bar{y} = x(\bar{p}, \tau)$  and then appeal to the uniqueness of the polar decomposition to prove that  $F_\tau(\bar{y}, \tau) = U_\tau(\bar{y}, \tau) = R_\tau(\bar{y}, \tau) = I$ .**

Recall that  $F = \nabla x$ . Then,

$$F(\bar{p}, t) = \nabla_p x(\bar{p}, t) = \nabla_p x_\tau(x(\bar{p}, \tau), t)$$

using the chain rule, then,

$$\nabla_p x_\tau(x(\bar{p}, \tau), t) = \nabla_y x_\tau(\bar{y}, t) \nabla_p x(\bar{p}, \tau) = F_\tau(\bar{y}, t)F(\bar{p}, \tau)$$

so that  $F(\bar{p}, t) = F_\tau(\bar{y}, t)F(\bar{p}, \tau)$ .

Setting  $t = \tau$ , I readily obtain  $F(\bar{p}, \tau) = F_\tau(\bar{y}, \tau)F(\bar{p}, \tau)$ . This obviously constrains that then  $F_\tau(\bar{y}, \tau) = I$ . Since the polar decomposition is unique, then, the obvious constraint

is then that if  $F = RU = I$ , then a solution is  $R = U = I$ . Since the polar decomposition is unique, then,  $F_\tau(\bar{y}, \tau) = U_\tau(\bar{y}, \tau) = R_\tau(\bar{y}, \tau) = I$ .

**c) Show that**  $C(\bar{p}, t) = F(\bar{p}, t)^T C_\tau(\bar{y}, t) F(\bar{p}, t)$ .

This is very straightforward to show using the fact that  $C = F^T F$  and the identity from part (b).

$$\begin{aligned} C(\bar{p}, t) &= F(\bar{p}, t)^T F(\bar{p}, t) \\ (\text{use } F(\bar{p}, t) &= F(\bar{y}, t) F(\bar{p}, \tau) \text{ from (b)}) \\ &= F(\bar{p}, \tau)^T [F(\bar{y}, t)^T F(\bar{y}, t)] F(\bar{p}, \tau) \\ &= F(\bar{p}, \tau)^T [C_\tau(\bar{y}, t)] F(\bar{p}, \tau) \end{aligned}$$

**d) Show that**

$$\begin{aligned} L(\bar{y}, \tau) &= \left. \frac{\partial}{\partial t} F_\tau(\bar{y}, t) \right|_{t=\tau} = \left[ \left. \frac{\partial}{\partial t} U_\tau(\bar{y}, t) + \frac{\partial}{\partial t} R_\tau(\bar{y}, t) \right] \right|_{t=\tau} \\ D(\bar{y}, \tau) &= \left. \frac{\partial}{\partial t} U_\tau(\bar{y}, t) \right|_{t=\tau} \\ W(\bar{y}, \tau) &= \left. \frac{\partial}{\partial t} R_\tau(\bar{y}, t) \right|_{t=\tau} \end{aligned}$$

By definition,  $L_m = \dot{F}F^{-1}$ . Further,  $L = D + W$  is the unique decomposition of  $L$  into its symmetric part,  $D$  and its antisymmetric part  $W$ .

Now,

$$\begin{aligned} L(\bar{p}, t) &= \dot{F}(\bar{p}, t) F(\bar{p}, t)^{-1} = \frac{\partial}{\partial t} [F_\tau(\bar{y}, t) F(\bar{p}, \tau)] [F_\tau(\bar{y}, t) F(\bar{p}, \tau)]^{-1} \\ &= \frac{\partial}{\partial t} [F_\tau(\bar{y}, t)] F_\tau(\bar{y}, t)^{-1} \end{aligned}$$

so that evaluating at  $\bar{y} = \bar{x}$  and  $t = \tau$ ,

$$L(\bar{y}, \tau) = \left. \frac{\partial}{\partial t} [F_\tau(\bar{y}, t)] F_\tau(\bar{y}, t)^{-1} \right|_{t=\tau} = \left. \frac{\partial}{\partial t} [F_\tau(\bar{y}, t)] \right|_{t=\tau}.$$

Next,

$$\left[ \frac{\partial}{\partial t} F \right]_{t=\tau} = \left[ \frac{\partial}{\partial t} RU \right]_{t=\tau} = \left[ \left[ \frac{\partial}{\partial t} R \right] U + R \left[ \frac{\partial}{\partial t} U \right] \right]_{t=\tau}$$

However, in part (b) I determined that  $F_\tau(\bar{y}, \tau) = U_\tau(\bar{y}, \tau) = R_\tau(\bar{y}, \tau) = I$ . Then,

$$\left[ \left[ \frac{\partial}{\partial t} R \right] U + R \left[ \frac{\partial}{\partial t} U \right] \right]_{t=\tau} = \left[ \frac{\partial}{\partial t} R \right]_{t=\tau} + \left[ \frac{\partial}{\partial t} U \right]_{t=\tau} .$$

In this case,  $U$  is a Hermitian positive-definite matrix in the reals, and so is symmetric. In order for  $U$  to remain symmetric at all times, then, the derivative of  $U$  must itself be symmetric.

Further, in this case  $R$  a rotation tensor with determinant 1. In order for  $R$  to retain this property at all times, then, its derivative must be skew.

Now, since the decomposition  $L = D + W$  is unique and

$$L(\bar{y}, \tau) = \frac{\partial}{\partial t} F_\tau(\bar{y}, t) \Big|_{t=\tau} = \left[ \frac{\partial}{\partial t} U_\tau(\bar{y}, t) + \frac{\partial}{\partial t} R_\tau(\bar{y}, t) \right]_{t=\tau}$$

$$\left[ \frac{\partial}{\partial t} R \right]_{t=\tau} \in \text{skew} \quad \left[ \frac{\partial}{\partial t} U \right]_{t=\tau} \in \text{symmetric}$$

Then it must be the case that

$$D(\bar{y}, \tau) = \frac{\partial}{\partial t} U_\tau(\bar{y}, t) \Big|_{t=\tau}$$

$$W(\bar{y}, \tau) = \frac{\partial}{\partial t} R_\tau(\bar{y}, t) \Big|_{t=\tau}$$

e) Show that  $\frac{\partial^{n+2}}{\partial t^{n+2}} F_\tau(\bar{y}, t) \Big|_{t=\tau} = \text{grad } a^{(n)}(\bar{y}, \tau)$  where  $a^{(n)}$  is the spatial description of the material time derivative of  $x$  of order  $n + 2$ .

Here,  $a$  represents acceleration, e.g.  $\text{grad } a^{(n)}(\bar{y}, \tau) = \text{grad } x^{(n+2)}(\bar{y}, \tau)$ .

Then,

$$\text{grad } x^{(n+2)}(\bar{y}, \tau) = \text{grad } \left[ \frac{\partial^{n+2}}{\partial t^{n+2}} x(\bar{y}, t) \right]_{t=\tau} = \frac{\partial^{n+2}}{\partial t^{n+2}} [\text{grad } x(\bar{y}, t)]_{t=\tau} = \frac{\partial^{n+2}}{\partial t^{n+2}} F(\bar{y}, t) \Big|_{t=\tau} .$$

**(Gurtin 9.4) Let  $x$  be a  $C^\infty$  motion. The tensors  $A_n(y, \tau) = \frac{\partial^n}{\partial t^n} C_\tau(y, \tau) \Big|_{t=\tau}$  where  $n = (1, 2, \dots)$  are called the Rivlin-Ericksen tensors.**

a) Show that  $A_1 = 2D$ .

Recall that  $C = U^2$ . Then,

$$A_1 = \left[ \frac{\partial}{\partial t} C \right]_{t=\tau} = \left[ \frac{\partial}{\partial t} U^2 \right]_{t=\tau} = \left[ \left[ \frac{\partial}{\partial t} U \right] U + U \left[ \frac{\partial}{\partial t} U \right] \right]_{t=\tau}.$$

However, from problem 9.1 above, I recall that

$$\left[ \frac{\partial}{\partial t} U \right]_{t=\tau} = D \quad \text{and} \quad [U]_{t=\tau} = I$$

So that:

$$A_1 = \left[ \left[ \frac{\partial}{\partial t} U \right] U + U \left[ \frac{\partial}{\partial t} U \right] \right]_{t=\tau} = 2D.$$

**b) Show that  $C^{(n)} = F^T A_n F$ , where  $C^{(n)}$  is the  $n$ th material time derivative of  $C$ .**

Using the result from 9.1c above, I have:

$$C(\bar{p}, t) = F(\bar{p}, \tau)^T C_\tau(\bar{y}, t) F(\bar{p}, \tau)$$

and

$$\begin{aligned} C^{(n)}(\bar{p}, t) &= \frac{\partial^n}{\partial t^n} C(\bar{p}, t) = \frac{\partial^n}{\partial t^n} F(\bar{p}, \tau)^T C_\tau(\bar{y}, t) F(\bar{p}, \tau) \\ &= F(\bar{p}, \tau)^T \left[ \frac{\partial^n}{\partial t^n} C_\tau(\bar{y}, t) \right] F(\bar{p}, \tau) = F(\bar{p}, \tau)^T A_n(\bar{y}, t) F(\bar{p}, \tau) \end{aligned}$$

**c) Verify that  $A_{n+1} = \dot{A}_n + A_n L + L^T A_n$ .**

Taking the result from 9.4b,

$$C^{(n)} = F^T A_n F$$

Because  $\dot{F} = LF$ ,

$$C^{(n+1)} = \dot{F}^T A_n F + F^T \dot{A}_n F + F^T A_n \dot{F} = F^T L^T A_n F + F^T \dot{A}_n F + F^T A_n L F$$

But,  $C^{(n+1)} = F^T A_{n+1} F$

Then,

$$F^T A_{n+1} F = F^T L^T A_n F + F^T \dot{A}_n F + F^T A_n L F$$

$$F^T A_{n+1} F = F^T [L^T A_n + \dot{A}_n + A_n L] F$$

$$A_{n+1} = L^T A_n + \dot{A}_n + A_n L$$

**(Gurtin 11.2)**

**Use the notation:**

$$w = \text{curl } v, \quad v = (\det F)_s, \quad L = \text{grad } v$$

**Establish the identity  $(vw)^{\bullet} = vLw + v(\text{curl } \dot{v})$ . Note that when  $\dot{v}$  is the gradient of a potential this reduces to  $(vw)^{\bullet} = vLw$ .**

Start with the identity

$$\dot{v} = v' + \frac{1}{2} \text{grad } (v^2) + (\text{curl } v) \times v$$

and

$$\dot{v} = (\det F)^{\bullet} = (\det F) \text{tr}(\dot{F}F^{-1}) = (\det F)(\text{tr } L)$$

Then, using the fact that curl is linear and that the curl of a gradient is zero,

$$(\text{curl } \dot{v}) = (\text{curl } v') + \frac{1}{2} (\text{curl } (\text{grad } (v^2))) + \text{curl } ((\text{curl } v) \times v)$$

$$(\text{curl } \dot{v}) = (\text{curl } v') + \text{curl } ((\text{curl } v) \times v)$$

Next, using  $\text{curl } (w \times v) = (\text{grad } w)v - (\text{grad } v)w + (\text{div } v)w - (\text{div } w)v$ ,

$$(\text{curl } \dot{v}) = [(\text{curl } v') + (\text{grad } (\text{curl } v))v] - (\text{grad } v)(\text{curl } v) + (\text{div } (\text{curl } v))v - (\text{div } v)(\text{curl } v)$$

$$(\text{curl } \dot{v}) = [\text{curl } v]^{\bullet} - Lw + (\text{tr } L)w$$

e.g.,

$$Lw = (\text{tr } L)w$$

Now I return to the identity

$$\dot{v} = (\det F)^{\bullet} = (\det F) \text{tr}(\dot{F}F^{-1}) = (\det F)(\text{tr } L)$$

and also use

$$\dot{w} = (\text{curl } v)^{\bullet} = \text{curl } \dot{v}$$

So that now:

$$\begin{aligned} (vw)^{\bullet} &= \dot{v}w + v\dot{w} \\ &= (\det F)(\text{tr } L)w + v(\text{curl } \dot{v}) \\ &= v[(\text{tr } L)w] + v(\text{curl } \dot{v}) \\ &= vLw + v(\text{curl } \dot{v}) \end{aligned}$$

Naturally, if  $\dot{v}$  is the gradient of a potential, the curl of the gradient is zero and this reduces to  $(vw)^{\bullet} = vLw$ .

**(Gurtin 11.5)** Let  $v$  be the gradient of a potential  $\varphi$ . Show that  $\dot{v} = \text{grad} \left( \varphi' + \frac{v^2}{2} \right)$

so that the acceleration is also the gradient of a potential.

$$v = \text{grad } \varphi$$

$$\dot{v} = v' + (\text{grad } v)v = (\text{grad } \varphi)' + (\text{grad } v)v$$

However, recall the well-known identity:

$$\text{grad } (a \cdot b) = (\text{grad } a)b + (\text{grad } b)a + a \times (\text{curl } b) + b \times (\text{curl } a)$$

and therefore,

$$\text{grad } (v \cdot v) = \text{grad } (v^2) = (\text{grad } v)v + (\text{grad } v)v + v \times (\text{curl } v) + v \times (\text{curl } v).$$

Since the curl of a gradient is zero and  $v$  is the gradient of a potential  $\varphi$ , however, the previous equation simplifies to:

$$\frac{1}{2} \text{grad } v^2 = (\text{grad } v)v.$$

Now I see, using my result above that:

$$\dot{v} = (\text{grad } \varphi)' + (\text{grad } v)v = (\text{grad } \varphi)' + \left( \text{grad } \frac{v^2}{2} \right) = \text{grad} \left( \varphi' + \frac{v^2}{2} \right).$$

**Calculate  $l(P,0)$ ,  $a(P,0)$ , and  $\alpha(P,0)$  with respect to a part  $P$  about the origin  $o$  at time  $t = 0$ , e.g., the linear momentum, angular momentum, and center of mass respectively for the following shearing motion  $\chi$ :**

$$\chi(p,t) = p + \gamma e_1 \otimes e_2 [p - o]$$

**for every  $p \in E$  and  $t \in \text{Reals}$ . Here,  $\gamma \in \text{Reals}$ ,  $e_1, e_2$  perpendicular unit vectors in  $V$ , and  $o \in E$  are given, and the part  $P$  is specified by the condition:**

$$P_0 := \chi(P,0) := \{x \in E \mid 0 \leq (x - o) \cdot e_i \leq 1, i = 1,2,3\}$$

**with  $e_3 := e_1 \times e_2$ . Finally, the density field  $\rho(\cdot,0)$  is taken to be a constant with value  $d > 0$ . Because  $\chi(p,0) = p$  for all  $p \in P_0$ , the integrals in the definitions of  $l(P,0)$ ,  $a(P,0)$ , and  $\alpha(P,0)$  are taken over  $P_0$ , the variable of integration  $x$  becomes  $p$ , and the position vector  $r(x) = x - o$  can be written  $r(p) = p - o$ .**

$P_0$  is, at time zero, the unit cube into the positive quadrant from the origin.

Taking care of the easiest part first, then,

$$m(P,0) = \int_{P_0} \rho dV = \int_0^1 \int_0^1 \int_0^1 ddr_1 dr_2 dr_3 = d$$

$$\alpha(P,0) - o = \frac{1}{m(P,0)} \int_{P_0} r \rho dV = \frac{1}{m(P,0)} \int_0^1 \int_0^1 \int_0^1 (r_1 e_1 + r_2 e_2 + r_3 e_3) ddr_1 dr_2 dr_3 = \frac{1}{2} (e_1 + e_2 + e_3)$$

For the other parts, I first need to determine the velocity corresponding to

$$\chi(p,t) = p + \gamma e_1 \otimes e_2 [p - o].$$

$$v(p,t) = \frac{\partial}{\partial t} \chi(p,t) = \gamma e_1 \otimes e_2 [p - o] = \gamma ([p - o] \cdot e_2) e_1.$$

Next, the linear velocity with respect to the origin is:

$$l(P,0) = \int_{P_0} v \rho dV = \int_0^1 \int_0^1 \int_0^1 \gamma (r \cdot e_2) e_1 ddr_1 dr_2 dr_3 = \gamma d e_1 \int_0^1 \int_0^1 r_2 dr_1 dr_2 dr_3 = \frac{\gamma d}{2} e_1$$

Finally, the angular velocity of the part about the origin is:

$$\begin{aligned} a(P,0) &= \int_{P_0} (r \times v) \rho dV = \int_0^1 \int_0^1 \int_0^1 [r \times \gamma (r \cdot e_2) e_1] ddr_1 dr_2 dr_3 \\ &= \gamma d \int_0^1 \int_0^1 \int_0^1 r_2 (r \times e_1) dr_1 dr_2 dr_3 \\ &= \gamma d \int_0^1 \int_0^1 \int_0^1 r_2 (-r_2 e_3 + r_3 e_2) dr_1 dr_2 dr_3 \\ &= \gamma d \left[ e_2 \int_0^1 \int_0^1 \int_0^1 r_2 r_3 dr_1 dr_2 dr_3 - e_3 \int_0^1 \int_0^1 \int_0^1 r_2^2 dr_1 dr_2 dr_3 \right] \\ &= \gamma d \left[ \frac{e_2}{4} - \frac{e_3}{3} \right] \end{aligned}$$