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Continuum Mechanics Homework 2

Gurtin 4.3) Establish the component representations:

$$(\nabla \varphi)_i = \frac{\partial \varphi}{\partial x_i} \quad (\nabla v)_{ij} = \frac{\partial v_i}{\partial x_j} \quad div \quad v = \sum_i \frac{\partial v_i}{\partial x_i}$$

$$(div \quad S)_i = \sum_j \frac{\partial S_{ij}}{\partial x_j} \quad \Delta\varphi = \sum_i \frac{\partial^2 \varphi}{\partial x_i^2} \quad (\Delta v)_j = \Delta v_j$$

These can be established from the relation

$$D\Phi(x)[e_i] = \frac{\partial \Phi(x)}{\partial x_i}$$

The definition of gradient is:

$$D\Phi(x)[u] = \nabla\Phi(x) \cdot u$$

Then,

$$(\nabla \varphi(x))_i = D\varphi(x)[e_i] = \frac{\partial \varphi(x)}{\partial x_i}$$

$$(\nabla v)_{ij} = e_i \cdot Dv[e_j] = \left(D\left(\sum_k v_k e_k \right)[e_j] \right) \cdot e_i = \sum_k (e_k \otimes \nabla v_k)[e_j] \cdot e_i$$

$$= \sum_k (\nabla v_k \cdot e_j) e_k \cdot e_i = \sum_k \frac{\partial v_k}{\partial x_j} \delta_{ki} = \sum_k \frac{\partial v_i}{\partial x_j}$$

The definition of divergence is:

$$div \quad v = tr \quad \nabla v$$

So that using the divergences determined above, I have:

Invoking an earlier relation,

$$div \quad v = \sum_i (\nabla v)_{ii} = \sum_i \frac{\partial v_i}{\partial x_i}$$

$$(div \quad S)_i = e_i \cdot (div \quad S) = div \quad (S^T[e_i]) = div \quad \left(\sum_j (S_{ij} e_j) \right)$$

$$= \sum_j div \quad (S_{ij} e_j) = \sum_j e_j \cdot \nabla S_{ij} = \sum_j \frac{\partial S_{ij}}{\partial x_j}$$

The definition of Laplacian is:

$$\Delta \Phi = div \quad \nabla \Phi$$

Invoking some earlier relations,

$$\Delta \varphi = div \quad \nabla \varphi = \sum_i \frac{\partial}{\partial x_i} \left(\frac{\partial \varphi}{\partial x_i} \right) = \sum_i \frac{\partial^2 \varphi}{\partial x_i^2}$$

Invoking earlier relations again,

$$(\Delta v)_i = (div \quad \nabla v)_i = \sum_j \frac{\partial}{\partial x_j} (\nabla v)_{ij} = \sum_j \frac{\partial}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} \right) = \sum_j \frac{\partial^2 v_i}{\partial x_j^2}$$

Gurtin 4.10) Let u and v be smooth. Show that
$div \quad (u \times v) = v \cdot (curl \quad u) - u \cdot (curl \quad v)$**.**

I will take a Tensor-notation approach to this proof, for simplicity and conciseness:

$$(u \times v)_i \equiv \varepsilon_{ijk} u^j v^k$$

$$div \quad u \equiv \partial_i u^i$$

$$(curl \quad u)_i \equiv \varepsilon_{ijk} \partial_j v^k$$

$$\therefore div \quad u \times v = \partial_i \varepsilon_{ijk} u^j v^k = \varepsilon_{ijk} \left[(\partial_i u^j) v^k + u^j (\partial_i v^k) \right]$$

$$= \varepsilon_{ijk} (\partial_i u^j) v^k + \varepsilon_{ijk} u^j (\partial_i v^k) = v^i \varepsilon_{ijk} (\partial_j u^k) - u^i \varepsilon_{ijk} (\partial_j v^k)$$

$$= v \cdot (curl \quad u) - u \cdot (curl \quad v)$$

Gurtin 5.1) Let R be a bounded region, and let $v, w : R \rightarrow V$ and $S : R \rightarrow Lin$ be smooth. Show that:

(a) $\int_{\partial R} (v \otimes n) dA = \int_R \nabla v dV$

Using the fact that the unit normal and the area element necessarily point in the same direction,

$$\left[\int_{\partial R}(v \otimes n)dA\right]r = \int_{\partial R}((n \cdot r)vdA) = \int_{\partial R} n_i r_i v_j dA = \left[\int_{\partial R} n_i v_j dA\right]r_i$$

Now using the divergence theorem,

$$\left[\int_{\partial R} n_i v_j dA\right]r_i = \left[\int_R \partial_i v_j dV\right]r_i = \left[\int_R \nabla v dV\right]_{ij} r_i$$

$$\therefore \int_{\partial R}(v \otimes n)dA = \int_R \nabla v dV$$

Note that the gradient of a vector-valued function gives a tensor-valued function, so that $\left[\int_R \nabla v dV\right]_{ij}$ is a tensor.

(b) $\int_{\partial R}(Sn \otimes v)dA = \int_R \left[(div \quad S) \otimes v + S\nabla v^T\right]dV$

Let $a \in V$ be given. Then,

$$\left[\int_{\partial R}(Sn \otimes v)dA\right]a = \int_{\partial R}(Sn \otimes v)adA = \int_{\partial R}((a \cdot v)S)ndA$$

$$= \int_R (div \quad ((a \cdot v)S))dV$$

$$define \quad \varphi = a \cdot v$$

$$= \int_R (S\nabla \varphi + \varphi div S)dV$$

$$but,$$

$$\nabla(a \cdot v) = (\nabla v)^T a$$

$$(a \cdot v)(div \quad S) = ((div \quad S) \otimes v)a$$

$$\therefore \left[\int_{\partial R}(Sn \otimes v)dA\right]a = \left[\int_R \left[(div \quad S)(v \cdot r) + S(\nabla v^T)\right]dV\right]a$$

But, since $a \in V$ is arbitrary,

$$\int_{\partial R}(Sn \otimes v)dA = \int_R \left[(div \quad S) \otimes v + S\nabla v^T\right]dV .$$

Gurtin 5.2) Let v be a smooth vector field on an open region R. Show that $\int_{dP} v \cdot ndA = 0$ for every regular region $P \subset R$ if and only if $div \quad v = 0$.

Let $div \quad v = 0$. Then, using the divergence theorem, $\int_{dP} v \cdot n dA = \int_{P} div \quad v\, dV$. However, since $div \quad v = 0$, $\int_{dP} v \cdot n dA = \int_{P} 0 dV = 0$.

Let $\int_{dP} v \cdot n dA = 0$ for every regular region $P \subset R$. Then, by localization, for any closed sphere Ω_δ of radius δ, I have:

$$div \quad v = \lim_{\delta \to 0} \frac{1}{vol(\Omega_\delta)} \int_{\Omega_\delta} div \quad v\, dV = \lim_{\delta \to 0} \frac{1}{vol(\Omega_\delta)} \int_{d\Omega_\delta} v \cdot n dA = 0.$$ QED.

Gurtin 6.8) A deformation of the form

$$x_1 = f_1(p_1, p_2)$$
$$x_2 = f_2(p_1, p_2)$$
$$x_3 = p_3$$

is called a plane strain. Show that for such a deformation the principal stretch λ_3 (in the p_3 direction) is unity. Show further that the deformation is isochoric if and only if the other two principal stretches, λ_α and λ_β, satisfy $\lambda_\alpha = \frac{1}{\lambda_\beta}$.

Let $x = f(p)$. Then clearly,

$$F(x) = \nabla f_p(x) = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} & 0 \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly, then, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of $F = \nabla f_p(x)$ with eigenvalue $\lambda_3 = 1$, and so must also be an eigenvector of $C(x)$ and $U(x)$ with the same eigenvalue. This transformation is isochoric by definition if $\det \quad U = 1$. Since the determinant of a matrix is a product of the eigenvalues and $\lambda_3 = 1$, then the other principal stretches λ_α and λ_β must satisfy $\lambda_\alpha \lambda_\beta = 1$ in order for the transformation to be isochoric.

Gurtin 6.10) Establish the following analogs of the relation given below:

$$\int_{\partial f(P)} \varphi(x) m(x) dA_x = \int_{\partial P} \varphi(f(p)) G(p) n(p) dA_p$$
$$G = (\det F) F^{-T}$$

Where m and n are respectively the outward unit normal fields on $\partial f(P)$ and ∂P. Here, v and T are continuous fields on $f(B)$ with v vector-valued and T tensor-valued.

$$\int_{\partial f(P)} T(x)m(x)dA_x = \int_{\partial P} T(f(p))G(p)n(p)dA_p$$

Let $a \in V$ be given. Then,

$$a \cdot \int_{\partial f(P)} T(x)m(x)dA_x = \int_{\partial f(P)} a \cdot T(x)m(x)dA_x = \int_{\partial f(P)} T(x)^T a \cdot m(x)dA_x$$

From the given relation above, then,

$$a \cdot \left(\int_{\partial f(P)} T(x)m(x)dA_x \right) = \int_{\partial f(P)} T(x)^T a \cdot m(x)dA_x = \int_{\partial P} T(f(p))^T a \cdot G(p)n(p)dA_p$$

$$= a \cdot \int_{\partial P} T(f(p))G(p)n(p)dA_p$$

However, since $a \in V$ is arbitrary, then, I have

$$\int_{\partial f(P)} T(x)m(x)dA_x = \int_{\partial P} T(f(p))G(p)n(p)dA_p$$

$$\int_{\partial f(P)} (x-o)\times T(x)m(x)dA_x = \int_{\partial P} (f(p)-o)\times T(f(p))G(p)n(p)dA_p$$

Following the same strategy as in the last relation, let $a \in V$:

$$Let \quad v(x) = x - o$$

$$a \cdot \int_{\partial f(P)} v(x)\times T(x)m(x)dA_x$$

$$= \int_{\partial f(P)} a \cdot v(x)\times T(x)m(x)dA_x$$

$$= \int_{\partial f(P)} a \times v(x)\cdot T(x)m(x)dA_x$$

$$= \int_{\partial f(P)} T(x)^T (a \times v(x))m(x)dA_x$$

Using the relation given in the problem statement, then, I have:

$$= \int_{\partial P} T(f(p))^T (a \times v(f(p))) \cdot G(p) n(p) dA_p$$
$$= \int_{\partial P} (a \times v(f(p))) \cdot T(f(p)) G(p) n(p) dA_p$$
$$= \int_{\partial P} a \cdot (v(f(p)) \times T(f(p))) G(p) n(p) dA_p$$
$$= a \cdot \int_{\partial P} (v(f(p)) \times T(f(p))) G(p) n(p) dA_p$$

And since $a \in V$ is arbitrary, then I have:

$$\int_{\partial f(P)} (x - o) \times T(x) m(x) dA_x = \int_{\partial P} (f(p) - o) \times T(f(p)) G(p) n(p) dA_p$$

Gurtin 7.1) Show that if $f_\varepsilon (0 < \varepsilon < \varepsilon_0)$ is a one-parameter family of deformations and $|\nabla u_\varepsilon| = \varepsilon$, then

$$E_\varepsilon = U_\varepsilon - I + o(\varepsilon)$$
$$\det F_\varepsilon - 1 = div \quad u_\varepsilon + o(\varepsilon)$$

Give a physical interpretation of $\det F_\varepsilon - 1$ in terms of the volume change in the deformation f_ε.

Recall:

$$E_\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$$
$$F = I + \nabla u$$
$$C = U^2 = 1 + \nabla u + \nabla u^T + \nabla u^T \nabla u$$
$$B = V^2 = 1 + \nabla u + \nabla u^T + \nabla u \nabla u^T$$

Now, if $C = U^2$, then $DC(0)[S] = 2DU(0)[S]$ for all tensors S. U should be differentiable because C is composed of differentiable deformations u. Now expanding about zero, I have:

$$U(0)[S] = I + DU(0)[S] + ... = 1 + \frac{1}{2} DC(0)[S] + ... = 1 + \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T \nabla u) + ...$$
$$= 1 + \frac{1}{2}(2E + o(\varepsilon)) + ... = 1 + E + o(\varepsilon)$$

Next, take

$$\det F_\varepsilon - 1 = \det(I + \nabla u) - 1 = 1 + tr\nabla u + o(\nabla u) - 1 = tr\nabla u + o(\nabla u) = div \quad u + o(\varepsilon)$$

If, for a physical system, the operator corresponding to F, $f(p) = p + u(p)$, moves each position as $u(p)$, then $\det F - 1$ gives the infinitesimal volume ratio $\frac{V_f - V}{V}$, indicating how much the final volume allocated to the initial infinitesimal volume about which this transformation is valid has changed.

Gurtin 7.3) Define the mean strain $\overline{E}$ by $vol(B)\overline{E} = \int_B E dV$. Show that $vol(B)\overline{E} = \frac{1}{2}\int_{\partial B}(u \otimes n + n \otimes u)dA$, so that $\overline{E}$ depends only on the boundary values of u.

Recall that: $E = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$.

$$vol(B)\overline{E} = \int_B E dV$$
$$= \frac{1}{2}\int_B \left(\nabla u + \nabla u^T\right) dV$$
$$= \frac{1}{2}\int_B \nabla u dV + \frac{1}{2}\int_B \nabla u^T dV$$
$$= \frac{1}{2}\int_B \nabla u dV + \frac{1}{2}\left(\int_B \nabla u dV\right)^T$$

As shown in problem 5.1a above, then,

$$= \frac{1}{2}\int_{\partial B} u \otimes n dA + \frac{1}{2}\left(\int_{\partial B} u \otimes n dA\right)^T$$
$$= \frac{1}{2}\int_{\partial B}(u \otimes n + n \otimes u)dA$$

QED.

Gurtin 7.4) Let $W = \frac{1}{2}\left(\nabla u - \nabla u^T\right)$. Show that:

$$|E|^2 + |W|^2 = |\nabla u|^2$$
$$|E|^2 - |W|^2 = \nabla u \cdot \nabla u^T$$

Recall that $E = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$.

$$|E|^2 = \frac{1}{2}\left(\nabla u + \nabla u^T\right)\cdot \frac{1}{2}\left(\nabla u + \nabla u^T\right) = \frac{1}{4}\left(\nabla u \cdot \nabla u + 2\nabla u \cdot \nabla u^T + \nabla u^T \cdot \nabla u^T\right)$$

$$= \frac{1}{2}\left(\nabla u \cdot \nabla u + \nabla u \cdot \nabla u^T\right)$$

$$|W|^2 = \frac{1}{2}\left(\nabla u - \nabla u^T\right)\cdot \frac{1}{2}\left(\nabla u - \nabla u^T\right) = \frac{1}{4}\left(\nabla u \cdot \nabla u - 2\nabla u \cdot \nabla u^T + \nabla u^T \cdot \nabla u^T\right)$$

$$= \frac{1}{2}\left(\nabla u \cdot \nabla u - \nabla u \cdot \nabla u^T\right)$$

Then,

$$|E|^2 + |W|^2 = \nabla u \cdot \nabla u = |\nabla u|^2$$

$$|E|^2 - |W|^2 = \nabla u \cdot \nabla u^T$$

Gurtin 7.5) (Korn's Inequality) Let u be of class C^2 and suppose that $u = 0$ on ∂B. Show that $\int_B |\nabla u|^2\, dV \le 2\int_B |E|^2\, dV$.

It can be shown that for any class C^2 vector field that

$$\nabla u \cdot \nabla u^T = div\left[(\nabla u)u - (div \quad u)u\right] + (div \quad u)^2.$$

$$\int_B |\nabla u|^2\, dV = \int_B \left[2|E|^2 - \nabla u \cdot \nabla u^T\, dV\right]$$

Further:

$$\nabla u \cdot \nabla u^T = div\left[(\nabla u)u - (div \quad u)u\right] + (div \quad u)^2$$

$$\int_B |\nabla u|^2\, dV = \int_B \left[2|E|^2 - div\left[(\nabla u)u - (div \quad u)u\right] - (div \quad u)^2\, dV\right]$$

$$= 2\int_B |E|^2\, dV - \int_B (div \quad u)^2\, dV - \int_B div\left[(\nabla u)u - (div \quad u)u\right] dV$$

$$= 2\int_B |E|^2\, dV - \int_B (div \quad u)^2\, dV - \int_{\partial B} \left[(\nabla u)u - (div \quad u)u\right]\cdot n\, dA$$

Since $u = 0$ on ∂B, then:

$$= 2\int_B |E|^2\, dV - \int_B (div \quad u)^2\, dV$$

But, $(div \quad u)^2$ is strictly positive so that then

$$\int_B |\nabla u|^2\, dV \ge 2\int_B |E|^2$$