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Continuum Mechanics Homework 1

Gurtin 1.6) Prove that:

a) $S(a \otimes b) = (Sa \otimes b)$

Consider the action of this operator on an arbitrary vector v.
By definition of tensor product,
$(a \otimes b)v = (b \cdot v)a$
so that
$S(a \otimes b)v = S(b \cdot v)a = (b \cdot v)Sa$.
Again invoking the definition of tensor product,
$(b \cdot v)Sa \rightarrow (Sa \otimes b)v$. QED.

b) $(a \otimes b)S = (a \otimes S^T b)$

Consider again the action of this operator on an arbitrary vector v.
By definition of tensor product,
$(a \otimes b)v = (b \cdot v)a$
so that
$(a \otimes b)Sv = (b \cdot Sv)a$
by the definition of transpose,
$b \cdot Sv = S^T b \cdot v$
so that
$(b \cdot Sv)a = (S^T b \cdot v)a$
By definition of tensor product,
$(S^T b \cdot v)a = (a \otimes S^T b)v$. QED.

c) $\sum_i (Se_i) \otimes e_i = S$

For any set of basis vectors, $\sum_i e_i \otimes e_i = I$.

Using the identity shown in part (a), I see that
$\sum_i (Se_i) \otimes e_i = \sum_i S(e_i \otimes e_i) = S \sum_i (e_i \otimes e_i) = S$

Gurtin 1.14) Let $\varphi : V \times V \times V \rightarrow R$ be trilinear and skew symmetric; that is, φ is linear in each argument and $\varphi(u,v,w) = -\varphi(v,u,w) = -\varphi(w,v,u) = -\varphi(u,w,v)$ for all $u,v,w \in V$. Let $S \in Lin$. Show that $\varphi(Se_1,e_2,e_3) + \varphi(e_1,Se_2,e_3) + \varphi(e_1,e_2,Se_3) = (tr \quad S)\varphi(e_1,e_2,e_3)$.

Using the identity $Se_1 = \sum_j S_{j1} e_j$, I see that then:

$$\begin{aligned}
&\varphi(Se_1, e_2, e_3) \\
&= \varphi\left(\sum_j S_{j1} e_j, e_2, e_3\right) \\
&= \varphi(S_{11}e_1, e_2, e_3) + \varphi(S_{21}e_2, e_2, e_3) + \varphi(S_{31}e_3, e_2, e_3) \\
&= S_{11}\varphi(e_1, e_2, e_3) + S_{21}\varphi(e_2, e_2, e_3) + S_{31}\varphi(e_3, e_2, e_3)
\end{aligned}$$

However, by skew symmetry,

$\varphi(e_2, e_2, e_3) = -\varphi(e_2, e_2, e_3)$ (interchanging the first and arguments)

so that $\varphi(e_2, e_2, e_3) = -\varphi(e_2, e_2, e_3) = 0$, and similarly for $\varphi(e_3, e_2, e_3)$. Then,

$$\begin{aligned}
&\varphi(Se_1, e_2, e_3) \\
&= S_{11}\varphi(e_1, e_2, e_3) + S_{21}\varphi(e_2, e_2, e_3) + S_{31}\varphi(e_3, e_2, e_3) \\
&= S_{11}\varphi(e_1, e_2, e_3) + 0 + 0 = S_{11}\varphi(e_1, e_2, e_3)
\end{aligned}$$

An analogous procedure can be done on the other two terms, giving:

$$\begin{aligned}
&\varphi(Se_1, e_2, e_3) + \varphi(e_1, Se_2, e_3) + \varphi(e_1, e_2, Se_3) \\
&= S_{11}\varphi(e_1, e_2, e_3) + S_{22}\varphi(e_1, e_2, e_3) + S_{33}\varphi(e_1, e_2, e_3) \\
&= (tr \quad S)\varphi(e_1, e_2, e_3)
\end{aligned}$$

Gurtin 1.15) Let Q be an orthogonal tensor, and let e be a vector with $Qe = e$.
(a) Show that $Q^T e = e$

$$\begin{aligned}
&Qe = e \\
&Q^T Qe = Q^T e \\
&because \quad Q \in Orth \quad Q^T Q = QQ^T = I \\
&e = Q^T e
\end{aligned}$$

(b) Let w be the axial vector corresponding to the skew part of Q. Show that w is parallel to e.

Decomposing Q into a symmetric part E and an antisymmetric part W,

$Q = E + W$

$E = \frac{1}{2}\left(Q + Q^T\right) \quad W = \frac{1}{2}\left(Q - Q^T\right)$

Then,

$$We = \frac{1}{2}\left(Q - Q^T\right)e = \frac{1}{2}\left(Qe - Q^T e\right) = \frac{1}{2}(e - e) = 0$$

However, since the axial vector is defined as the vector $w \in V$ such that $w \times v = Wv \quad \forall \quad v \in V$, so that $Wv = 0$ for vectors parallel to the axial vector, $We = 0$ implies $w \parallel e$.

Gurtin 2.1) Determine the spectrum, the characteristic spaces, and a spectral decomposition for each of the following tensors:

$A = \alpha I + \beta m \otimes m$

$B = m \otimes n + n \otimes m$

Here, α and β are scalars, while m and n are orthogonal unit vectors.

$$A = \alpha I + \beta m \otimes m = (\alpha + \beta) m \otimes m + \alpha\left(I - m \otimes m\right)$$

This implies that this tensor has two eigenvalues in its spectrum, α and $\alpha + \beta$. These correspond to spaces $\{e\}^{\perp}$ and $\{e\}$, respectively.

The spectral decomposition is then: $S = (\alpha + \beta) m \otimes m + \alpha\left(m^{\perp 1} \otimes m^{\perp 1}\right) + \alpha\left(m^{\perp 2} \otimes m^{\perp 2}\right)$, where $m^{\perp 1}$ and $m^{\perp 2}$ are two unit vectors orthogonal to one another and m.

Now consider $B = m \otimes n + n \otimes m$.

For any vector in this space, $Bv = (n \cdot v)m + (m \cdot v)n$.

By inspection, I see that valid eigenvectors (characteristic spaces) and their corresponding eigenvalues are:

$$e_1 = \frac{n + m}{\|n + m\|}, \lambda_1 = 1 \quad e_2 = \frac{n - m}{\|n + m\|}, \lambda_2 = -1 \quad e_3 = n \times m, \lambda_3 = 0$$

The spectral decomposition is then:

$$S = \sum_i \lambda_i \left(e_i \otimes e_i\right),$$

with λ_i, e_i defined as above.

Gurtin 2.3) Let $D \in Sym$, $Q \in Orth$. Show that the spectrum of D equals the spectrum of QDQ^T. Show further that if e is an eigenvector of D, then Qe is an eigenvector of QDQ^T corresponding to the same eigenvalue.

Define $e' = Qe$.

Then, since $Q \in Orth$ $\quad Q^T Q = QQ^T = I$ and $De = \lambda e$,

$QDQ^T e' = QDQ^T Qe = QDe = \lambda Qe = \lambda QQ^T e' = \lambda e'$, indicating that the new eigenvectors $e' = Qe$ have the same eigenvalues and thus the same spectrum as the eigenvectors e of D. The same argument in reverse shows that the spectra are in fact the same.

Gurtin 2.6) Let $F = RU$ and $F = VR$ denote the right and left polar decompositions of $F \in Lin^+$.
(a) Show that U and V have the same spectrum $(\omega_1, \omega_2, \omega_3)$.

Since $F = RU$ and $F = VR$, then $U = R^T F = R^T VR$. From problem 2.3, I have already shown that this relation implies that U and V have the same spectrum since $R \in Orth$.

(b) Show that F and R admit the representations

$$F = \sum_i \omega_i (f_i \otimes e_i)$$
$$R = \sum_i f_i \otimes e_i$$

where e_i and f_i are respectively the eigenvectors of U and V corresponding to ω_i.

Again writing $U = R^T F = R^T VR$, I first demonstrate that $R = \sum_i f_i \otimes e_i$:

Now since e_i and f_i are orthonormal bases, then I may let $f_i = (R)e_i$.

Next, note that $R = R\sum_i e_i \otimes e_i = \sum_i (R)e_i \otimes e_i = \sum_i f_i \otimes e_i$,

Now I demonstrate that $F = \sum_i \omega_i (f_i \otimes e_i)$.

Since $F = VR$, $F = VR = V\left(\sum_i f_i \otimes e_i \right) = \sum_i Vf_i \otimes e_i = \sum_i \omega_i f_i \otimes e_i$. I have justified this operation in problem 1.6.

Gurtin 3.1) Compute $DG(A)$ for each of the following functions $G : Lin \to Lin$.

a) $G(A) = (tr \quad A)A$

$G(A+H) - G(A) = (tr \quad A+H)(A+H) - (tr \quad A)A = (tr \quad H)A + (tr \quad A)H + (tr \quad H)H$

$= (tr \quad A)A + (tr \quad H)A + o(H)$

$\therefore DG(A)[H] = (tr \quad H)A + (tr \quad A)H$

b) $G(A) = ABA$ **with** B **a given tensor.**

$G(A+H) - G(A) = (A+H)B(A+H) - ABA = ABH + HBA + HBH$

$= ABH + HBA + o(H)$

$\therefore DG(A)[H] = ABH + HBA$

c) $G(A) = A^T A$

$G(A+H) - G(A) = (A+H)^T (A+H) - A^T A = A^T H + H^T A + H^T H$

$= A^T H + H^T A + o(H)$

$\therefore DG(A)[H] = A^T H + H^T A$

d) $G(A) = (u \cdot Au)A$ **with** u **a given vector.**

$G(A+H) - G(A) = (u \cdot (A+H)u)(A+H) - (u \cdot Au)A$

$= (u \cdot Au)H + (u \cdot Hu)A + (u \cdot Hu)H$

$= (u \cdot Au)H + (u \cdot Hu)A + o(H)$

$\therefore DG(A)[H] = (u \cdot Au)H + (u \cdot Hu)A$

Gurtin 3.2) Let G be defined on the set of all invertible tensors by $G(A) = A^{-1}$. Assuming that G is differentiable, show that $G(A)[H] = -A^{-1}HA^{-1}$.

$Define \quad G(A) = A \quad F(A) = A^{-1}$

Then,

$AA^{-1} = I$

$[DG(A)[H]]A^{-1} + A[DF(A)[H]] = 0$

$HA^{-1} + A[DF(A)[H]] = 0$

$A[DF(A)[H]] = -HA^{-1}$

$DF(A)[H] = -A^{-1}HA^{-1}$

Gurtin 3.6) Let $Q : R \to Orth$ be differentiable. Show that $Q(t)\dot{Q}(t)^T$ is skew at each $t \in R$.

$$QQ^T = I$$
$$[D(Q)[\alpha]]Q^T + Q[D(Q^T)[\alpha]] = 0$$
$$\alpha\dot{Q}Q^T + \alpha Q\dot{Q}^T = 0$$
$$\dot{Q}Q^T = -Q\dot{Q}^T$$
$$\dot{Q}Q^T = -(\dot{Q}Q)^T$$
$$(skew \quad by \quad definition \quad of \quad skew: A^T = -A)$$

Additional Problem #1:

Let $f : \mathrm{E} \to \mathrm{E}$ **be defined by** $f(x) = x + ((x - x_0) \cdot e_2)^2 e_1$ **for all** $x \in \mathrm{E}$**, where** $e_1, e_2 \in V$ **and** $x_0 \in \mathrm{E}$ **are given and satisfy** $e_1 \cdot e_1 = 1 \quad e_2 \cdot e_2 = 1 \quad e_1 \cdot e_2 = 0$**. Find** $Df(x)$ **in each of the following ways:**

(a) Using the definition of derivative.

$$f(x+v) - f(x) = (x+v) + ((x+v-x_0)\cdot e_2)^2 e_1 - x - ((x-x_0)\cdot e_2)^2 e_1$$
$$= v + \left[2(v\cdot e_2)((x-x_0)\cdot e_2) + (v\cdot e_2)^2\right] e_1$$
$$= v + 2(v\cdot e_2)((x-x_0)\cdot e_2) e_1 + o(v)$$

(b) By computing the Jacobian matrix of a coordinate version $\hat{f}$ **of** f **in a Cartesian coordinate system in which** x_0 **is the origin and** e_1, e_2 **are members of the orthonormal basis used to define the system.**

Take
$$v \leftrightarrow \vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \quad x - x_0 \leftrightarrow \vec{x} = x_x\hat{x} + x_y\hat{y} + x_z\hat{z}$$
$$x_0 \leftrightarrow \vec{0}$$
$$e_1 \leftrightarrow \hat{x} \quad e_2 \leftrightarrow \hat{y}$$

Here, v_x, v_y, v_z are defined to be scalar functions corresponding to components of a vector parameter.

Further, x_x, x_y, x_z are defined to be scalar functions corresponding to components of a Cartesian coordinate parameter.

$$f(x) = x + ((x - x_0)\cdot e_2)^2 e_1$$
$$\hat{f}(x) = x_x\hat{x} + x_y\hat{y} + x_z\hat{z} + \left[(x_x\hat{x} + x_y\hat{y} + x_z\hat{z})\cdot\hat{y}\right]^2\hat{x} = x_x\hat{x} + x_y\hat{y} + x_z\hat{z} + {x_y}^2\hat{x}$$
$$J = \begin{bmatrix} \frac{\partial\hat{f}\cdot\hat{x}}{\partial x_x} & \frac{\partial\hat{f}\cdot\hat{x}}{\partial x_y} & \frac{\partial\hat{f}\cdot\hat{x}}{\partial x_z} \\ \frac{\partial\hat{f}\cdot\hat{y}}{\partial x_x} & \frac{\partial\hat{f}\cdot\hat{y}}{\partial x_y} & \frac{\partial\hat{f}\cdot\hat{y}}{\partial x_z} \\ \frac{\partial\hat{f}\cdot\hat{z}}{\partial x_x} & \frac{\partial\hat{f}\cdot\hat{z}}{\partial x_y} & \frac{\partial\hat{f}\cdot\hat{z}}{\partial x_z} \end{bmatrix} = \begin{bmatrix} 1 & 2x_y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$J\vec{v} = \vec{v} + 2x_y(\vec{v}\cdot\hat{y})\hat{x}$$
$$Df(x)[v] = v + 2((x - x_0)\cdot e_2)(v\cdot e_2)e_1$$

Additional Problem #2:

Let $Q \in Orth^+$ **be given.**

(a) Show that for every $v_1, v_2 \in V$ **with** $v_1 \neq v_2$**,** $(Qv_1 = v_1, Qv_2 = v_2, Q \neq I)$ **implies that** v_1, v_2 **are linearly dependent.**

Suppose that the statement above is false. I will show that then it is impossible that $Q = I$.

If $Q \in Orth^+$, then this operation preserves lengths and inner products, so that if I define an orthonormal basis $e_1, e_2, e_1 \times e_2$ corresponding to linearly independent v_1, v_2:

$$e_1 = \frac{v_1}{|v_1|} = \frac{Qv_1}{|Qv_1|} = f_1$$
$$e_2 = \frac{v_2}{|v_2|} = \frac{Qv_2}{|Qv_2|} = f_2$$
$$e_1 \times e_2 = \frac{v_1 \times v_2}{|v_1 \times v_2|} = \frac{Q(v_1 \times v_2)}{|Q(v_1 \times v_2)|}$$

However, since $e_1 = f_1, e_2 = f_2$, then it must be the case that $e_1 \times e_2 = \pm f_1 \times f_2$ since $Q \in Orth^+$ preserves lengths and inner products. The matrix for the transformation is then

$$Q_{\{e_1, e_2, e_1\times e_2\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

Since $Q \in Orth^{+}$, then $\det Q = 1$ which forces the lower-right entry to be 1, so that then $Q = I$ and this theorem is proven by contradiction.

(b) Show that Q has the number 1 in its spectrum.

$$\textit{because} \quad \det Q = 1$$
$$Q \quad \textit{is} \quad 3 \times 3$$
$$\det(Q - I) = \det(Q - I)^T = \det(Q^T - I) = \det Q \det(Q^T - I) = \det(QQ^T - Q) = \det(I - Q) = -\det(Q - I)$$
$$\therefore \det(Q - I) = 0$$
$$\det(Q - 1I) = 0$$

And so Q has 1 in its spectrum.