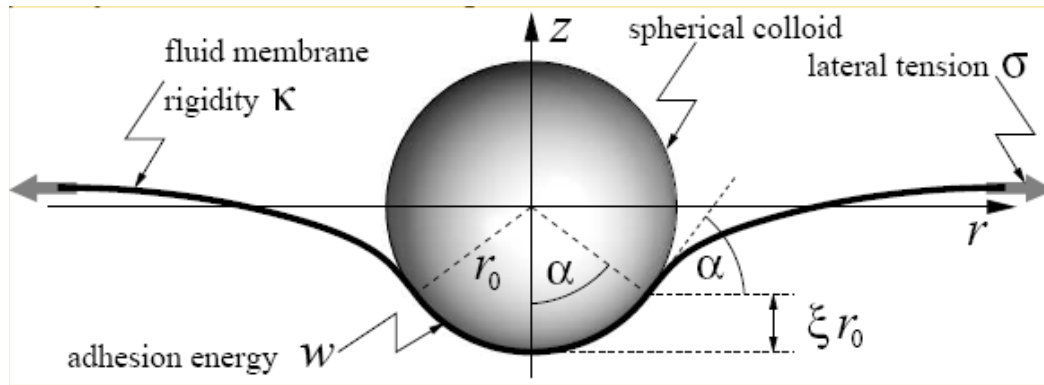


**11. Viral Budding, or: How a colloidal particle is wrapped by a fluid membrane**

A spherical colloid of radius  $r_0$  approaches a fluid membrane. Let us assume that the surface of this colloid is such that it would like to adhere to the membrane. Specifically, let there be a binding energy per area  $\omega$ , which means that an adhesion patch of area  $A$  lowers the total energy of the system by  $E_{ad} = -\omega A$ . What we now would like to find out is under which conditions the membrane would indeed begin to wrap around the colloid.



**1. Show that**

- (i) The dimensionless parameter  $\xi$  in the figure is given by  

$$\xi = 1 - \cos \alpha .$$

Using simple geometry, the vertical distance from the center of the particle to the lowest point of the dimple is  $r_0$ . The vertical distance from the center of the particle to the depth where the particle is not in contact with the membrane is  $r_0 \cos \alpha$ . Then, the difference between these is  $r_0 - r_0 \cos \alpha \equiv \xi r_0$  so that  $\xi = 1 - \cos \alpha$ .

- (ii) The area of adhesion between colloid and membrane is  

$$A = 2\pi r_0^2 \xi .$$

Integrating over the bowl-shaped surface where the colloid particle is in direct contact with the membrane:

$$A = \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} 2\pi(r_0 \cos \theta)r_0 d\theta = 2\pi r_0^2 \left[ \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2} - \alpha\right) \right] = 2\pi r_0^2 [1 - \cos \alpha] = 2\pi r_0^2 \xi$$

- (iii) This area is bigger than its horizontal projection by  

$$\Delta A = A - A_{||} = \pi r_0^2 \xi^2$$

Projecting the cup-shaped area up onto a plane perpendicular to the z-axis, I have

$$A_{\parallel} = \pi(r_0 \sin \alpha)^2 = \pi r_0^2 (1 - \cos^2 \alpha)$$

Then,

$$\Delta A = A - A_{\parallel} = 2\pi r_0^2 (1 - \cos \alpha) - \pi r_0^2 (1 - \cos^2 \alpha) = \pi r_0^2 (1 - 2\cos \alpha + \cos^2 \alpha) = \pi r_0^2 (1 - \cos \alpha)^2 = \pi r_0^2 \xi^2$$

- 2. At the bound patch we have bending energy (we neglect the Gaussian term), adhesion energy (proportional to  $A$ ). Show that the total energy of the bound region,  $E_{bound}$ , is given by**

$$\frac{E_{bound}}{\pi k} = (4 - 2\tilde{\omega})\xi + \tilde{\sigma}\xi^2$$

**with the dimensionless variables  $\tilde{\omega} = \frac{\omega r_0^2}{\kappa}$  and  $\tilde{\sigma} = \frac{\sigma r_0^2}{\kappa}$ .**

I recall that  $E_{bending} = 2\kappa c^2$ , where  $c = \frac{1}{r_0}$  is the curvature of the sphere.

$$E_{bound} = E_{ad} + E_{tension} + E_{bending} = -\omega A + \sigma \Delta A + \frac{2\kappa}{r_0^2} A = -\omega(2\pi r_0^2 \xi) + \sigma(\pi r_0^2 \xi^2) + \frac{2\kappa}{r_0^2}(2\pi r_0^2 \xi)$$

Then,

$$\frac{E_{bound}}{\pi \kappa} = -2 \frac{\omega r_0^2}{\kappa} \xi + \frac{\sigma r_0^2}{\kappa} \xi^2 + 4\xi = (4 - 2\tilde{\omega})\xi + \tilde{\sigma}\xi^2$$

- 3. Let us at first completely ignore the energy  $E_{free}$  of the non-adhering membrane. How does the form of the function  $E_{bound}(\xi)$  between  $\xi = 0$  (no wrapping) and  $\xi = 2$  (full wrapping) depend on the values of the system parameters? Under which condition does the particle wrap partially? Under which condition does it wrap completely? In a diagram with axes  $\tilde{\omega}$  and  $\tilde{\sigma}$  show the regions indicating “no wrapping”, “partial wrapping”, and “full wrapping”.**

$$\frac{E_{bound}}{\pi \kappa} = (4 - 2\tilde{\omega})\xi + \tilde{\sigma}\xi^2$$

Finding the local minimum of binding energy,

$$\frac{1}{\pi\kappa} \frac{\partial E_{bound}}{\partial \xi} = (4 - 2\tilde{\omega}) + 2\tilde{\sigma}\xi$$

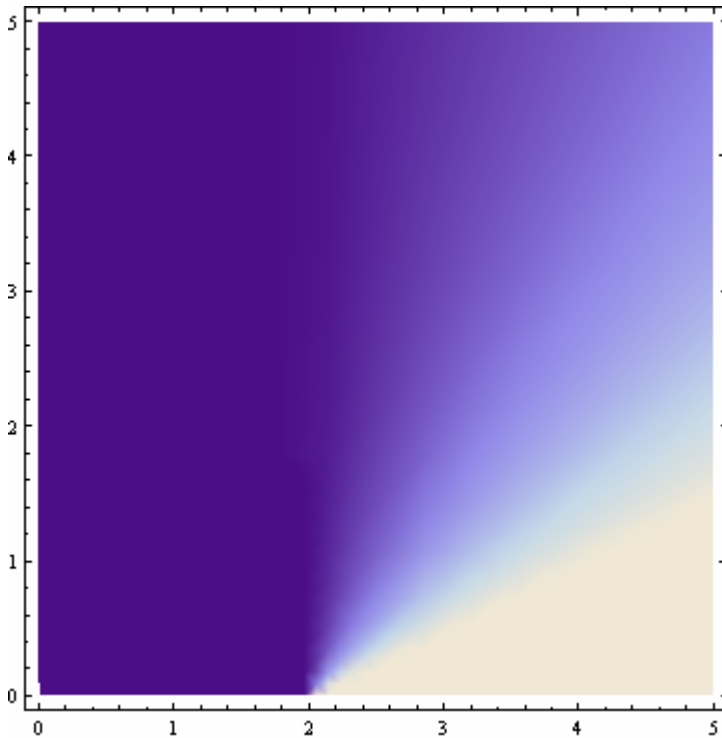
Then,

$$\frac{2\tilde{\omega} - 4}{2\tilde{\sigma}} = \xi_{preferred}$$

This corresponds to the minimum energy,

$$E_{bound, min} = \pi\kappa \left[ \frac{(4 - 2\tilde{\omega})^2}{4\tilde{\sigma}} - \frac{(4 - 2\tilde{\omega})^2}{2\tilde{\sigma}} \right] = \frac{\pi\kappa(4 - 2\tilde{\omega})^2}{4\tilde{\sigma}}$$

Plotting binding with  $\tilde{\omega}$  on the x-axis and  $\tilde{\sigma}$  on the y-axis,



Binding does not occur when  $\tilde{\omega} < 2$ , and the colloid is completely enveloped when  $\frac{2\tilde{\omega} - 4}{2\tilde{\sigma}} \geq 2$ . Above, the darkest region is unbound and the lightest region is completely bound.

## 12. Viral budding, continued...

We now want to find the shape and energy of the non-adhering membrane from problem 11, i.e. the function  $h(r)$  and its energy  $E_{free}$ . As you have

shown in problem 10, this function satisfies the differential equation

$\Delta(\Delta - \lambda^{-2})h = 0$ , with  $\lambda = \sqrt{\frac{\kappa}{\sigma}}$ . This equation is solved by the eigenfunctions

of the Laplacian to eigenvalues of 0 and  $\lambda^{-2}$ . For the given planar

axisymmetric situations these are (i) a constant, (ii)  $\ln\left(\frac{r}{\lambda}\right)$ , (iii)  $K_0\left(\frac{r}{\lambda}\right)$ , and

(iv)  $I_0\left(\frac{r}{\lambda}\right)$  (those two have eigenvalues  $\lambda^{-2}$ ). The last two are modified

Bessel functions of the second kind.

1. For  $r \rightarrow \infty$  the membrane should approach a horizontal plane. Also, we must satisfy the contact boundary conditions  $h(r_0 \sin \alpha) = -r_0 \cos \alpha$  and  $h'(r_0 \sin \alpha) = \tan \alpha$ . Using  $K_0'(x) = -K_1(x)$ , show that this gives the equilibrium shape

$$h(r) = -r_0 \cos \alpha + \frac{K_0\left(\frac{r_0}{\lambda} \sin \alpha\right) - K_0\left(\frac{r}{\lambda}\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \lambda \tan \alpha \quad (\text{for } r \geq r_0 \sin \alpha)$$

First, I would like to show that this satisfies the shape equation  $\Delta(\Delta - \lambda^{-2})h = 0$ . Recall

that  $\Delta K_0\left(\frac{r}{\lambda}\right) = \lambda^{-2} K_0\left(\frac{r}{\lambda}\right)$ .

$$\begin{aligned} & \Delta(\Delta - \lambda^{-2}) \left[ -r_0 \cos \alpha + \frac{K_0\left(\frac{r_0}{\lambda} \sin \alpha\right) - K_0\left(\frac{r}{\lambda}\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \lambda \tan \alpha \right] \\ &= \Delta \left[ \frac{-\lambda^{-2} K_0\left(\frac{r}{\lambda}\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \lambda \tan \alpha + \lambda^{-2} r_0 \cos \alpha - \lambda^{-2} \frac{K_0\left(\frac{r_0}{\lambda} \sin \alpha\right) - K_0\left(\frac{r}{\lambda}\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \lambda \tan \alpha \right] \\ &= \left[ \frac{-\lambda^{-4} K_0\left(\frac{r}{\lambda}\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \lambda \tan \alpha + \frac{\lambda^{-4} K_0\left(\frac{r}{\lambda}\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \lambda \tan \alpha \right] \\ &= 0 \end{aligned}$$

Thus I see that the above expression satisfies the shape equation. Next, verifying:

$$h(r_0 \sin \alpha) = -r_0 \cos \alpha + \frac{K_0\left(\frac{r_0}{\lambda} \sin \alpha\right) - K_0\left(\frac{r_0}{\lambda}\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \lambda \tan \alpha = -r_0 \cos \alpha$$

Finally, verifying:

$$h'(r_0 \sin \alpha) = \frac{\frac{1}{\lambda} K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \lambda \tan \alpha = \tan \alpha$$

- 2. The energy  $E_{free}$  of the non-adhering membrane is given by the surface integral of  $\frac{1}{2} \kappa (\Delta h)^2 + \frac{1}{2} \sigma (\nabla h)^2$  over all points with a distance bigger than  $r_0 \sin \alpha$  from the origin. The equilibrium energy of the free membrane is of course obtained by using the solution to part (1) above when doing this integral. For this, please remember that**

$$\Delta K_0\left(\frac{r}{\lambda}\right) = \lambda^{-2} K_0\left(\frac{r}{\lambda}\right). \text{ Using also the fact that}$$

$$\int_a^\infty dx x [K_0^2(x) + K_1^2(x)] = a K_0(a) K_1(a), \text{ show that:}$$

$$\frac{E_{free}}{\kappa \pi} = \frac{r_0 \sin^3(\alpha)}{\lambda \cos^2(\alpha)} \frac{K_0\left(\frac{r_0}{\lambda} \sin \alpha\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \cong -2 \tilde{\sigma} \xi^2 \ln \frac{\tilde{\sigma} \xi}{C} + O(\xi^3) \text{ with } C = \frac{2}{e^{2\gamma}} \approx 0.63047350\dots$$

First,

$$(\Delta h)^2 = \left( \frac{\tan \alpha}{\lambda} \frac{K_0\left(\frac{r}{\lambda}\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \right)^2$$

Next, using the polar gradient operator,  $\nabla = \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} + \frac{\partial}{\partial r} \hat{r}$ , I have:

$$h(r) = -r_0 \cos \alpha + \frac{K_0\left(\frac{r_0}{\lambda} \sin \alpha\right) - K_0\left(\frac{r}{\lambda}\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \lambda \tan \alpha$$

$$(\nabla h)^2 = \left( \frac{K_1\left(\frac{r}{\lambda}\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \tan \alpha \right)^2$$

Then,

$$\begin{aligned} & \frac{1}{2} \kappa (\Delta h)^2 + \frac{1}{2} \sigma (\nabla h)^2 \\ &= \frac{1}{2} \kappa \left( \frac{\tan \alpha}{\lambda} \frac{K_0\left(\frac{r}{\lambda}\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \right)^2 + \frac{1}{2} \sigma \left( \frac{K_1\left(\frac{r}{\lambda}\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)} \tan \alpha \right)^2 \\ &= \frac{\tan^2 \alpha}{2 K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)^2} \left[ \frac{\kappa}{\lambda^2} K_0\left(\frac{r}{\lambda}\right)^2 + \sigma K_1\left(\frac{r}{\lambda}\right)^2 \right] \\ &= \frac{\kappa \tan^2 \alpha}{2 \lambda^2 K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)^2} \left[ K_0\left(\frac{r}{\lambda}\right)^2 + K_1\left(\frac{r}{\lambda}\right)^2 \right] \end{aligned}$$

Next, I must integrate:

$$\begin{aligned}
E_{free} &= \int_0^{2\pi} d\theta \int_{r_0 \sin \alpha}^{\infty} r dr \left[ \frac{1}{2} \kappa (\Delta h)^2 + \frac{1}{2} \sigma (\nabla h)^2 \right] = 2\pi \int_{r_0 \sin \alpha}^{\infty} r dr \frac{\kappa \tan^2 \alpha}{2\lambda^2 K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)^2} \left[ K_0\left(\frac{r}{\lambda}\right)^2 + K_1\left(\frac{r}{\lambda}\right)^2 \right] \\
&= \frac{\pi \kappa \tan^2 \alpha}{\lambda^2 K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)^2} \int_{r_0 \sin \alpha}^{\infty} r dr \left[ K_0\left(\frac{r}{\lambda}\right)^2 + K_1\left(\frac{r}{\lambda}\right)^2 \right] = \frac{\pi \kappa \tan^2 \alpha}{\lambda K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)^2} \left[ r_0 \sin \alpha K_0\left(\frac{r_0}{\lambda} \sin \alpha\right) K_1\left(\frac{r_0}{\lambda} \sin \alpha\right) \right] \\
&= \frac{\pi \kappa \sin^3 \alpha}{\lambda \cos^2 \alpha} \frac{K_0\left(\frac{r_0}{\lambda} \sin \alpha\right)}{K_1\left(\frac{r_0}{\lambda} \sin \alpha\right)}
\end{aligned}$$

Just as claimed.

- 3. The total penetration follows from  $\frac{d(E_{bound} + E_{free})}{d\xi} = 0$ . Using the equation from part 1 above and the small- $\xi$ -approximation in part 2 above, show that for small  $\xi$  this condition – which cannot easily be solved for  $\xi$  – can be written as:**

$$\frac{2 - \tilde{\omega}}{C} = \Xi \ln \Xi \quad \text{with} \quad \Xi = \frac{\tilde{\sigma} \xi}{C}.$$

**Explain that, if nothing else, this shows at least that  $\xi \propto \frac{1}{\tilde{\sigma}}$  (i.e., penetration varies inversely with tension).**

$$E_{bound} + E_{free} = \pi \kappa (4 - 2\tilde{\omega}) \xi + \pi \kappa \tilde{\sigma} \xi^2 - 2\pi \kappa \tilde{\sigma} \xi^2 \ln \frac{\tilde{\sigma} \xi}{C}$$

Then,

$$\frac{d(E_{bound} + E_{free})}{d\xi} = \pi \kappa (4 - 2\tilde{\omega}) + 2\pi \kappa \tilde{\sigma} \xi - 4\pi \kappa \tilde{\sigma} \xi \ln \frac{\tilde{\sigma} \xi}{C} - 2\pi \kappa \xi \frac{\tilde{\sigma}^2}{C}$$

$$\text{Solving } \frac{d(E_{bound} + E_{free})}{d\xi} = 0,$$

$$0 = (4 - 2\tilde{\omega}) + 2\tilde{\sigma} \xi - 4\tilde{\sigma} \xi \ln \frac{\tilde{\sigma} \xi}{C} - 2\xi \frac{\tilde{\sigma}^2}{C} = (4 - 2\tilde{\omega}) + 2C\Xi - 4C\Xi \ln \Xi - 2\tilde{\sigma}\Xi$$

In the very small  $\xi$  limit, the  $\Xi \ln \Xi$  term dominates so that

$$0 = (4 - 2\tilde{\omega}) - 4C\Xi \ln \Xi$$

$$\frac{2 - \tilde{\omega}}{2C} = \Xi \ln \Xi$$

This disagrees by a factor of two from the claimed result, but at any rate still gives the expected proportionality.

Since the left-hand side of the above result is a property of the membrane independent of  $\xi$  and  $\tilde{\sigma}$ , this means that the right-hand side must also be constant. Since the right-hand side  $\Xi \ln \Xi$  is a function only of the dimensionless parameter  $\Xi = \frac{\tilde{\sigma}\xi}{C}$  and  $C$  is a

dimensionless constant, this implies that  $\xi \propto \frac{1}{\tilde{\sigma}}$  in order to hold  $\Xi$  and therefore both sides of the equation constant under a change in the tension parameter  $\tilde{\sigma}$ .