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Quantum Mechanics 3 Mid-Term

1) **<u>The Cosmic Microwave Background Radiation</u>: For the first 10,000 years after the Big Bang, the Universe was a bath of radiation in local equilibrium. Consider the statistical mechanics of a gas of free photons and obtain:**

i) **The pressure, the internal density $u = \frac{U}{V}$, and the equation of state $p = p(U)$.**

First, recall that the Boson distribution function for particles with no chemical potential (such as photons) in the Grand canonical Ensemble:

$$n(E) = \frac{1}{e^{\beta E} - 1}.$$

Further, I know that the momentum of a photon is given by:

$$p = \hbar k$$

and the energy by:

$$E = cp$$

Calculating the internal energy density, then I have:

$$u = \sum_{k,\lambda} En(E) \to 2\int \frac{d^3k}{(2\pi)^3} \frac{\hbar ck}{e^{\beta\hbar ck} - 1}$$

Here I have converted the sum into an integral in k-space; the factor of 2 indicates that for each direction of propagation, there are two independent choices of orientation of electric and magnetic field perpendicular to this direction.

Then:

$$Spherical \quad Symmetry: \quad d^3k \to 4\pi k^2 dk \quad Photon \quad Spin: \quad s = 1$$

$$u = \frac{1}{c^3\hbar^3\pi^2} \int_0^\infty \frac{(\hbar ck)^3}{e^{\beta\hbar ck} - 1} d(\hbar ck) = \frac{\pi^2}{15c^3\hbar^3\beta^4}$$

In terms of this Grand Canonical distribution, then, pressure is given by (Pathria):

$$P \equiv \frac{1}{\beta}\ln Z = -\sum_{k,\lambda}\ln\left(1-e^{-\beta\hbar ck}\right)$$

$$\to -\frac{1}{\hbar^3 c^3 \pi^2 \beta}\int_0^\infty (\hbar ck)^2 \ln\left(1-e^{-\beta\hbar ck}\right) d(\hbar ck)$$

$$= -\frac{1}{\hbar^3 c^3 \pi^2 \beta^4}\int_0^\infty (\beta\hbar ck)^2 \ln\left(1-e^{-\beta\hbar ck}\right) d(\beta\hbar ck)$$

Now I integrate by parts:

$$-\frac{1}{\hbar^3 c^3 \pi^2 \beta^4}\int_0^\infty X^2 \ln\left(1-e^{-X}\right) d(X)$$

$$u = \ln\left(1-e^{-X}\right) \qquad dv = X^2 dx$$

$$du = \frac{e^{-X}}{1-e^{-X}}dX \qquad v = \frac{1}{3}X^3$$

$$= -\frac{1}{\hbar^3 c^3 \pi^2 \beta^4}\left[\ln\left(1-e^{-\beta X}\right)X^3\Big|_{X=0}^{\infty} - \frac{1}{3}\int_0^\infty \frac{X^3}{1-e^{-X}}d(X)\right]$$

$$= \frac{1}{3\hbar^3 c^3 \pi^2 \beta^4}\int_0^\infty \frac{X^3}{1-e^{-\beta X}}d(X) = \frac{1}{3}\frac{\pi^2}{15\hbar^3 c^3 \beta^4} = \frac{1}{3}u$$

This incidentally also gives the equation of state:

$$P = \frac{1}{3}u$$

$$PV = \frac{1}{3}U$$

ii) In an expanding Universe, the total volume V becomes a function of time through the expansion as $V(t) = V_0\left(\frac{a(t)}{a_0}\right)^3$ **where** $a(t)$ **is a scale factor,** a_0 **is the scale factor today, and** $\frac{\dot{a}(t)}{a(t)} = H_c$ **is the Hubble "constant" where** $\dot{a}(t) = \frac{da(t)}{dt}$ **and** t **= time. Consider an adiabatic (isentropic) expansion so that** $\frac{dS}{dt} = 0$ **with** S **the total entropy of the photon gas. Show that** $\frac{dS}{dt} = 0$ **implies that the total number of photons is conserved, i.e.** $\frac{dN}{dt} = 0$**.**

Let me quickly calculate N.

$$n = \sum_{k,\lambda} n(E) \rightarrow 2\int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta\hbar ck} - 1}$$

$$2\int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta\hbar ck} - 1} = \frac{1}{\hbar^3 c^3 \pi^2} \int \frac{(\hbar ck)^2 d(\hbar ck)}{e^{\beta\hbar ck} - 1} = \frac{1}{\hbar^3 c^3 \pi^2} \frac{2\zeta(3)}{\beta^3}$$

Next I will calculate S :

Before going to the isentropic expansion, first consider an isothermal expansion so that I may obtain a function in terms of temperature for entropy:

At constant <u>temperature</u>, then,

$$\sigma \equiv \frac{\pi^2 k^4}{15 c^3 \hbar^3}$$

$$\Delta U = Q + \Delta W$$

$$W = -\int P(t) dV$$

$$From \quad part \quad (i), isothermal:$$

$$\Delta U = \sigma T^4 \Delta V = Q + \Delta W$$

$$W = -\frac{1}{3}\sigma T^4 \int dV = -\frac{1}{3}\sigma T^4 \Delta V$$

$$\therefore Q = \frac{4}{3}\sigma T^4 \Delta V$$

Then consulting Pathria for a formula relating this to entropy along an adiabat, I have:

$$\Delta S_{rev} = \frac{Q}{T} = \frac{4}{3}\sigma T^3 \Delta V \ .$$

Since with no volume I can have no particles and so no entropy, I need not concern myself with a constant term and so I may drop the deltas:

$$S = \frac{4}{3}\sigma T^3 V$$

Now, examining my relationships:

$$S = \frac{4}{3}\sigma T^3 V \ \text{ and } \ N = \frac{V}{\hbar^3 c^3 \pi^2} \frac{2\zeta(3)}{\beta^3},$$

I see that they both depend on volume and temperature in the same way! This indicates that if one is constant, then the other must also be.

iii) For this adiabatic expansion show that the temperature of the photon gas must depend on time as $T(t) = T_0 \frac{a_0}{a(t)}$.

If $\frac{dS}{dt} = 0$, then using the result from part (ii),

$$\frac{dS}{dt} = 0 = \frac{d}{dt}\left[\frac{4}{3}\sigma T^3 V\right] = \frac{4}{3}\sigma\left(3T^2\frac{dT}{dt}V + T^3\frac{dV}{dt}\right)$$

Now I see that in order to satisfy this:

$$(1) \quad -3\frac{dT}{dt}V = T\frac{dV}{dt}$$

(1) RHS :

$$T\frac{dV}{dt} = T\left(3V_0\left(\frac{a(t)}{a_0}\right)^2\frac{\dot{a}(t)}{a_0}\right) = -V(t)H_c$$

Verify this by the expected form: $T(t) = T_0 \frac{a_0}{a(t)}$

(1) LHS :

$$\frac{1}{T}V(t)\frac{dT}{dt} = \frac{1}{T_0}\frac{a(t)}{a_0}\left(-T_0\frac{a_0\dot{a}(t)}{a(t)^2}\right)V(t) = -\frac{\dot{a}(t)}{a(t)}V(t) = -V(t)H_c$$

So I see that the right-hand side matches the left-hand side of (1), just as expected!

2) <u>Melting of a Crystal</u>: Consider a crystal of N identical atoms of mass M in three dimensions, of volume V and lattice spacing a. Assume that there is only an acoustic phonon branch with $\omega(\vec{k}) = c|\vec{k}|$ for all wave vectors in the Brillouin Zone. A criterion for melting of the lattice (Lindmann's criterion) is that the lattice melts when the mean square fluctuation of the atomic displacement at temperature T is of the order of the square of the lattice spacing, namely: $\left.\langle q_l^2\rangle\right|_{T_M} = a^2$ where $\langle\ldots\rangle$ is the average in the canonical ensemble at temperature T and T_M is the melting temperature.

i) Assume a spherical Brillouin Zone so that for all $|\vec{k}| < k_{BZ}$ there is one branch of acoustic phonons with $\omega(\vec{k}) = c|\vec{k}|$ and there are N normal modes in the Brillouin Zone. Find k_{BZ}.

Taking a cube of this material, I have:

$$\frac{V}{N} = a^3$$

$$N = \sum_k^{k_{BZ}} \rightarrow \frac{V}{8\pi^3} \int_0^{k_{BZ}} d^3k = \frac{V}{6\pi^2} k_{BZ}^3$$

$$k_{BZ} = \frac{\left(6\pi^2\right)^{\frac{1}{3}}}{a}$$

ii) Obtain $\langle q_l^2 \rangle$ in the canonical ensemble (only one branch of acoustic phonons) and subtract the $T = 0$ value since this is the "zero point" motion.

The RMS displacement is given roughly by the Hamiltonian, as:

$$\left\langle q_l^{\,2}(\theta,\phi)\right\rangle \approx \sum_{k\in BZ} \frac{\left(n_k + \frac{1}{2}\right)\hbar}{\omega_k}$$

$$\omega_k = ck \quad n_k = \frac{e^{-\beta E(k)}}{\sum_{k\in BZ} e^{-\beta E(k)}} \quad E(k) = \hbar ck$$

$$\sum_{k\in BZ} e^{-\beta\hbar ck} \rightarrow \frac{V}{8\pi^3} \int_0^{k_{BZ}} e^{-\beta\hbar c|\vec{k}|} d^3k = \frac{V}{2\pi^2} \int_0^{k_{BZ}} e^{-\beta\hbar ck} k^2 dk = \frac{2 - \left(\left(\beta\hbar ck_{BZ}\right)^2 + 2\beta\hbar ck_{BZ} + 2\right) e^{-\beta\hbar ck_{BZ}}}{\left(\beta c\hbar\right)^3} = C_{norm}(\beta)$$

Then:

$$\left\langle q_l^{\,2}(\theta,\phi)\right\rangle \approx \sum_{k\in BZ} \frac{\left(n_k + \frac{1}{2}\right)\hbar}{\omega_k} \rightarrow \sum_{k\in BZ} \frac{\left(C_{norm}(\beta) e^{-\beta\hbar ck} + \frac{1}{2}\right)\hbar}{ck} \rightarrow \frac{V}{2\pi^2} \int_0^{k_{BZ}} \frac{\left(C_{norm}(\beta) e^{-\beta\hbar ck} + \frac{1}{2}\right)\hbar}{ck} k^2 dk$$

$$= \frac{\hbar k_{BZ}^2}{4c} + \frac{C_{norm}(\beta)\left(1 - \left(1 + \beta\hbar ck\right) e^{-\beta\hbar ck}\right)}{\beta^2 c^3 \hbar}$$

Or, subtracting off the zero-point term,

$$\left(\left\langle q_l^2 \right\rangle\Big|_T - \left\langle q_l^2 \right\rangle\Big|_{T=0}\right) = \frac{C_{norm}(\beta)\left(1 - \left(1 + \beta\hbar ck\right) e^{-\beta\hbar ck}\right)}{\beta^2 c^3 \hbar}$$

$$= \frac{2}{c^6 \hbar^4} e^{-2\frac{\hbar ck_{BZ}}{kT}} \left[e^{\frac{\hbar ck_{BZ}}{kT}} kT - kT - c\hbar k_{BZ} \right] \left[e^{\frac{\hbar ck_{BZ}}{kT}} (kT)^2 - (kT)^2 - c\hbar k_{BZ} kT - \frac{1}{2}\left(c\hbar k_{BZ}\right)^2 \right]$$

iii) **Assume that** $T_M >> \frac{\hbar c k_{BZ}}{k_B}$ **and obtain the melting temperature from** $\left(\langle q_l^2\rangle\big|_T - \langle q_l^2\rangle\big|_{T=0}\right) = a^2$**.**

Finally,

$$\beta \to 0, e^{-\beta\hbar c k_{BZ}} \to 1:$$

$$\left(\langle q_l^2\rangle\big|_T - \langle q_l^2\rangle\big|_{T=0}\right) \to \frac{2}{c^6\hbar^4}\left[c\hbar k_{BZ}\right]^2 T_M = a^2$$

$$T_M = \frac{a^2 c^4 \hbar^2}{2k_{BZ}^2}$$

iv) **Repeat part iii, but for the opposite limit** $T_M << \frac{\hbar c k_{BZ}}{k_B}$

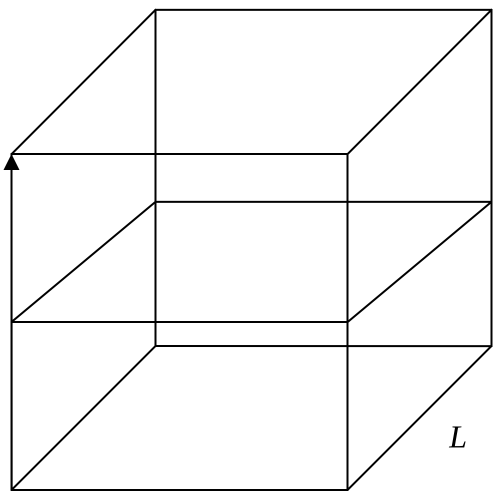
$$\beta \to \infty, e^{-\beta\hbar c k_{BZ}} \to 0$$

$$\left(\langle q_l^2\rangle\big|_T - \langle q_l^2\rangle\big|_{T=0}\right) = \frac{C_{norm}(\beta)\left(1-(1+\beta\hbar c k)e^{-\beta\hbar c k}\right)}{\beta^2 c^3 \hbar}$$

$$\to \frac{2}{c^6\hbar^4}(kT_M)^3 = a^2$$

$$T_M = \frac{c^2}{k}\left(\frac{a^2\hbar^4}{2}\right)^{\frac{1}{3}}$$

3) **<u>The Casimir Effect</u>: Although the zero point energy of the electromagnetic (quantized) field is unobservable, Casimir suggested that the difference between the zero point energy of the vacuum and that in the case of two conducting plates in the vacuum can be measured. Consider the case of a volume** V **to be a cube of side** L **and putting two conducting parallel plates at** $z = 0, R < L$**:**



The conducting plates at $z=0, z=R$ are extended in the xy-plane $0 \le x, y \le L$. The conducting plates imply that the electric field vanishes on these plates at all times; $\vec{E}(x,y,z=0)=0=\vec{E}(x,y,z=R)$.

a) **Quantize the electromagnetic vector potential in Coulomb gauge to obey the vanishing condition $\vec{E}(x,y,z=0)=0=\vec{E}(x,y,z=R)$ at all times.**

In the Coulomb gauge, $\vec{\nabla}\cdot\vec{A}=0$. Further, $\vec{E}=-\dfrac{\partial\vec{A}}{\partial t}$.

I see that the electromagnetic vector potential in free space is:

$$\vec{A}_{\lambda,\vec{k}}(\vec{x},t)=\frac{1}{\sqrt{V}}\vec{\varepsilon}_{\lambda}(\vec{k})\left[\alpha_{k,\lambda}e^{i\vec{k}\cdot\vec{x}}e^{-i\omega(\vec{k})t}+\alpha^{*}{}_{k,\lambda}e^{-i\vec{k}\cdot\vec{x}}e^{i\omega(\vec{k})t}\right]$$
$$\vec{k}\quad free\quad \vec{\varepsilon}_{\lambda}(\vec{k})\perp\vec{k}\quad \omega(\vec{k})=ck$$

However, it must be corrected to vanish on the boundaries $z=0, z=R$:

$$\vec{A}_{\lambda,\vec{k},n}(\vec{x},t)=\frac{1}{\sqrt{V}}\vec{\varepsilon}_{\lambda}(\vec{k})\left[\alpha_{k,\lambda}e^{ik_x x}e^{ik_y y}\sin\left(\frac{n\pi}{R}z\right)e^{-i\omega(\vec{k})t}-\alpha^{*}{}_{k,\lambda}e^{-ik_x x}e^{-ik_y y}\sin\left(\frac{n\pi}{R}z\right)e^{i\omega(\vec{k})t}\right]$$
$$\vec{k}=\left\langle k_x,k_y,\frac{n\pi}{R}\right\rangle\quad \vec{\varepsilon}_{\lambda}(\vec{k})\perp\vec{k}\quad \omega(\vec{k})=c|k|$$

Where the proper normalization is

$$\sqrt{\frac{2\omega(\vec{k})}{\hbar c^2}}\alpha_{k,\lambda} = \hat{a}_{k,\lambda}$$

and the Hamiltonian

$$H = \int d^3x\left[\frac{1}{2}\vec{E}(\vec{x},t)^2 + \frac{1}{2}\vec{B}(\vec{x},t)^2\right] = \sum_{k,\lambda}\hbar\omega(\vec{k})\left(\hat{a}^+{}_{k,\lambda}\hat{a}_{k,\lambda} + \frac{1}{2}\right)$$

The only difference between the Hamiltonian now is the allowed vectors $\vec{k}$ are not a continuum in along the z-axis. This is the same as simple harmonic oscillators, with zero-point energies well-known as:

$$E_{0:k,\lambda} = \frac{\hbar\omega(\vec{k})}{2}$$

I have taken these results from a previous homework assignment (Homework #5).

b) In the presence of parallel plates the zero-point energy of the quantized electromagnetic field has two components: E_{in}, the energy inside the parallel plates corresponding to the EM field quantized with vanishing boundary conditions at $z = 0, z = R$, and E_{out} the energy outside the parallel plates where $R \le z \le L$ and the EM field is quantized just as in the vacuum case. Consider E_{out} becomes an integral but E_{in} contains a discrete sum. In the presence of the parallel plates the zero point energy is E_{in} + E_{out}.

$$\sum_{k,\lambda} in \to L^2\sum_{\lambda=1}^{2}\int\int\frac{dk_x dk_y}{(2\pi)^2}\sum_{n=0}^{\infty}\ldots$$

$$\langle E_{in}\rangle = L^2\sum_{\lambda=1}^{2}\int\int\frac{dk_x dk_y}{(2\pi)^2}\sum_{n=0}^{\infty}g(n)\frac{\hbar}{2}\omega\left(\left\langle k_x, k_y, \frac{n\pi}{R}\right\rangle\right)$$

$$\therefore \frac{\langle E_{in}\rangle}{L^2} = \int\int\frac{dk_x dk_y}{(2\pi)^2}\left(\hbar\omega(\langle k_x, k_y, 0\rangle) + 2\sum_{n=1}^{\infty}\hbar\omega\left(\left\langle k_x, k_y, \frac{n\pi}{R}\right\rangle\right)\right)$$

Above, $g(n)$ indicates the degeneracy in terms of valid orientations.

$$\sum_{k,\lambda} out \to L^2(L-R)\sum_{\lambda=1}^{2}\int \frac{d^3k}{(2\pi)^3}\ldots$$

$$\langle E_{out}\rangle = L^2(L-R)\sum_{\lambda=1}^{2}\int \frac{d^3k}{(2\pi)^3}\frac{\hbar}{2}\omega(\vec{k})$$

$$\therefore \frac{\langle E_{out}\rangle}{L^2(L-R)} = \int \frac{d^3k}{(2\pi)^3}\hbar\omega(\vec{k})$$

c) The energy difference between the vacuum case and the case with parallel plates is $\Delta E = E_{in} + E_{out} - E_v$ where E_v is the zero point energy in the vacuum case. Use the Euler-MacLaurin summation formula:

$$\sum_{n=0}^{\infty} F(n) = \int_0^{\infty} F(x)dx + \frac{1}{2}F(0) - \frac{B_2}{2!}F'(0) - \frac{B_4}{4!}F'''(0) + \ldots \quad B_2 = \frac{1}{6} \quad B_4 = -\frac{1}{30}$$

Show that the energy difference per unit area $\frac{\Delta E}{L^2} = \frac{C}{R^3}$ and find the constant C. The force between the plates per unit area is $\frac{F}{L^2} = -\frac{\partial\left(\frac{\Delta E}{L^2}\right)}{\partial R}$. Find this force and show that it is attractive (a trivial result from $\frac{\Delta E}{L^2}$ but one that highlights that the vacuum energy difference can be measured as an attractive force. This Casimir force was first measured by M. Spaarney.

Now I may make use of the Euler-MacLaurin summation formula in order to simplify the infinite integrals from part (b).

Start by considering the energy inside the plates:

$$\frac{\langle E_{in}\rangle}{L^2} = \int\int \frac{dk_x dk_y}{(2\pi)^2}\left(\hbar\omega(\langle k_x, k_y, 0\rangle) + 2\sum_{n=1}^{\infty}\hbar\omega\left(\left\langle k_x, k_y, \frac{n\pi}{R}\right\rangle\right)\right)$$

Performing the integral area-wise, I have:

$$\frac{\langle E_{in}\rangle}{L^2} = \int \frac{kdk}{2\pi}\left(\frac{\hbar c}{2}k + 2\sum_{n=1}^{\infty}\frac{\hbar c}{2}\sqrt{k^2 + \left(\frac{n\pi}{R}\right)^2}\right) = \int \frac{kdk}{2\pi}\left(-\frac{\hbar c}{2}k + 2\sum_{n=0}^{\infty}\frac{\hbar c}{2}\sqrt{k^2 + \left(\frac{n\pi}{R}\right)^2}\right)$$

$$= \frac{\hbar c}{12\pi}\lim_{\lambda\to\infty}\left[-\lambda^3 + 2\sum_{n=0}^{\infty}\left(\left(\frac{n^2\pi^2}{R^2} + \lambda^2\right)^{\frac{3}{2}} - \left(\frac{n\pi}{R}\right)^3\right)\right]$$

Now using the Euler-MacLaurin summation formula:

$$= \frac{\hbar c}{12\pi}\lim_{\lambda\to\infty}\left[-\lambda^3 + 2\left(\int_0^\infty\left(\left(\frac{n^2\pi^2}{R^2}+\lambda^2\right)^{\frac{3}{2}} - \left(\frac{n\pi}{R}\right)^3\right)dn + \frac{\lambda^3}{2} - \frac{B_4}{4!}\left(\frac{-6\pi^3}{R^3}\right)\right)\right]$$

$$= \frac{\hbar c}{6\pi}\lim_{\lambda\to\infty}\left[\int_0^\infty\left(\left(\frac{n^2\pi^2}{R^2}+\lambda^2\right)^{\frac{3}{2}} - \left(\frac{n\pi}{R}\right)^3\right)dn + \frac{B_4}{4!}\left(\frac{6\pi^3}{R^3}\right)\right]$$

$$= \frac{\hbar c}{6\pi}\lim_{\lambda\to\infty}\left[\int_0^\infty\left(\left(\frac{n^2\pi^2}{R^2}+\lambda^2\right)^{\frac{3}{2}} - \left(\frac{n\pi}{R}\right)^3\right)dn\right] + \frac{\hbar c B_4}{2\cdot 4!}\left(\frac{2\pi^2}{R^3}\right)$$

Now I identify:

$$\frac{\hbar c}{6\pi}\lim_{\lambda\to\infty}\left[\int_0^\infty\left(\left(\frac{n^2\pi^2}{R^2}+\lambda^2\right)^{\frac{3}{2}} - \left(\frac{n\pi}{R}\right)^3\right)dn\right] \to R\int_0^\infty \frac{\hbar\omega(k)k^2 dk}{2\pi^2}$$

Then:

$$\Delta E = E_{in} + E_{out} - E_v$$
$$= RL^2\int\frac{k^2dk}{2\pi^2}\hbar\omega(\vec{k}) + L^2\frac{\hbar c B_4}{4!}\left(\frac{2\pi^2}{R^3}\right) + (L-R)L^2\int\frac{k^2dk}{2\pi^2}\hbar\omega(\vec{k}) - L^3\int\frac{k^2dk}{2\pi^2}\hbar\omega(\vec{k})$$
$$\frac{\Delta E}{L^2} = \frac{\hbar c B_4}{2\cdot 4!}\left(\frac{2\pi^2}{R^3}\right) = -\frac{\hbar c\pi^2}{720R^3}$$

and

$$\frac{F}{L^2} = -\frac{\partial\left(\frac{\Delta E}{L^2}\right)}{\partial R} = -\frac{\hbar c\pi^2}{240R^4},$$

so that the force is pushing the top plate towards the bottom plate.

4) **<u>Pi-Mesons and Nuclear Forces</u>: before the advent of the theory of strong interactions, Yukawa proposed that the interaction between nucleons was mediated by pions: these are bosonic mesons described by a Klein-Gordon field. Neutrons can be modeled to interact via Pi-mesons with the following second-quantized Hamiltonian:**

$$H = \sum_p \varepsilon(p)c^+_{p,\sigma}c_{p,\sigma} + \sum_k \hbar\omega(k)\left[a^+_k a_k + \frac{1}{2}\right] + \int d^3x\,\psi^+_N(\vec{x})\Pi(\vec{x})\psi_N(\vec{x}),$$ **where** $c^+_{p,\sigma}, c_{p,\sigma}$ **are creation-annihilation operators for neutrons,** a^+_k, a_k **those for**

pions and $\omega(k)=\sqrt{c^2k^2+\frac{m^2c^2}{\hbar^2}}$ **,** m **is the pion mass (about 140 MeV) and** $\Pi(\vec{x})=\frac{1}{\sqrt{V}}\sum_k\sqrt{\frac{\hbar c^2}{2\omega(k)}}\left[a_k e^{i\vec{k}\cdot\vec{x}}+a_k{}^+e^{-i\vec{k}\cdot\vec{x}}\right]$**.**

a) **Write the interaction Hamiltonian in** c^+,c,a^+,a **(namely find the matrix elements) and draw the different processes.**

First, in terms of a first-quantized distribution:

$$-\frac{\hbar^2\nabla^2}{2m}U_{p,\sigma}(\vec{x})=\varepsilon(p)U_{p,\sigma}(\vec{x})$$

Then $\psi_N(\vec{x})=\sum_{p,\sigma}c_{p,\sigma}U_{p,\sigma}(\vec{x})$, and the interaction Hamiltonian

$$H_I=\int d^3x\psi_N^+(\vec{x})\Pi(\vec{x})\psi_N(\vec{x})$$

$$=\sum_{p,\sigma,p',\sigma',k}\int d^3xc^+_{p',\sigma'}U^+_{p',\sigma'}(\vec{x})\left[\frac{1}{\sqrt{V}}\sqrt{\frac{\hbar c^2}{2\omega(k)}}\left[a_ke^{i\vec{k}\cdot\vec{x}}+a_k{}^+e^{-i\vec{k}\cdot\vec{x}}\right]\right]c_{p,\sigma}U_{p,\sigma}(\vec{x})$$

$$=\frac{1}{\sqrt{V}}\sum_{p,\sigma,p',\sigma',k}\left[c^+_{p',\sigma'}a_kc_{p,\sigma}V^{\sigma,\sigma'}_{p,p',k}+c^+_{p',\sigma'}a_k{}^+c_{p,\sigma}\tilde{V}^{\sigma,\sigma'}_{p,p',k}\right]$$

$$V^{\sigma,\sigma'}_{p,p',k}\equiv\int d^3xU^+_{p',\sigma'}(\vec{x})\left[\sqrt{\frac{\hbar c^2}{2\omega(k)}}e^{i\vec{k}\cdot\vec{x}}\right]U_{p,\sigma}(\vec{x})$$

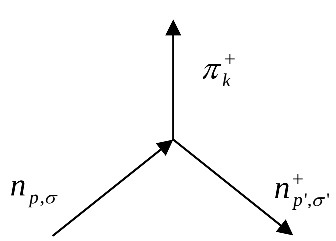
$$\tilde{V}^{\sigma,\sigma'}_{p,p',k}\equiv\int d^3xU^+_{p',\sigma'}(\vec{x})\left[\sqrt{\frac{\hbar c^2}{2\omega(k)}}e^{-i\vec{k}\cdot\vec{x}}\right]U_{p,\sigma}(\vec{x})$$

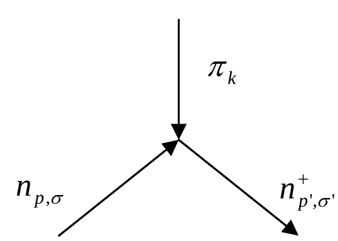
For a particle in free space, $U_{p,\sigma}(\vec{x})=e^{i\vec{p}\cdot\vec{x}}\chi_\sigma(\vec{x})$. Then:

$$V^{\sigma,\sigma'}_{p,p',k}=(2\pi)^3\sqrt{\frac{\hbar c^2}{2\omega(k)}}\delta_{\sigma,\sigma'}\delta^3\left(\frac{\vec{p}}{\hbar}+\vec{k}-\frac{\vec{p}'}{\hbar}\right)$$

$$\tilde{V}^{\sigma,\sigma'}_{p,p',k}=(2\pi)^3\sqrt{\frac{\hbar c^2}{2\omega(k)}}\delta_{\sigma,\sigma'}\delta^3\left(\frac{\vec{p}}{\hbar}-\vec{k}-\frac{\vec{p}'}{\hbar}\right)$$

These correspond to the neutron emitting or absorbing a pion:

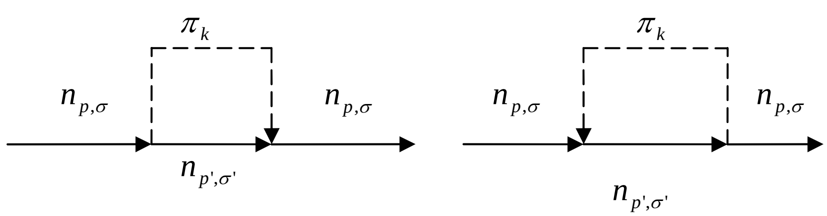




b) Compute the second-order shift in the energy of a neutron of momentum p; what are the intermediate states that contribute?

The first-order shifts should certainly be zero, since
$\varepsilon_{p,\sigma,k}^{(1)} = \langle k_\pi | \langle p,\sigma | H_I | p,\sigma \rangle | k_\pi \rangle$
is zero. Physically, this is true energetically since a neutron may not emit or absorbing a pion and remaining in the same state. Clearly, however, $\langle a \rangle = \langle a^+ \rangle = 0$.

The second-order shifts, however, correspond to these processes. For a neutron in the pion vacuum, only the emitted and re-absorbed situation is possible, and it is the same pion that is emitted and re-absorbed in this situation (a "virtual process").



Above I have omitted the creation symbol. However, I see that for the neutron vacuum that the second option is forbidden by causality.

The shift will be given, generally, by:

$$\varepsilon_{p,\sigma,k}^{(2)} = \sum_{p',\sigma',k} \frac{\left| \langle k'_\pi | \langle p',\sigma' | H_I | p,\sigma \rangle | k_\pi \rangle \right|^2}{\varepsilon(p,k) - \varepsilon(p',k') + i\eta}$$

I will assume that this neutron exists in the pion vaccum. Then:

$$\varepsilon_{p,\sigma}^{(2)} \equiv \varepsilon_{p,\sigma,0}^{(2)} = \sum_{p',\sigma',k'} \frac{\left| \langle k'_\pi | \langle p',\sigma' | H_I | p,\sigma \rangle | 0_\pi \rangle \right|^2}{\varepsilon(p,0) - \varepsilon(p',k') + i\eta}$$

Now I see that on substitution of the relevant parts, I have (defining $\varepsilon(p,k)$ as the energy of a state with neutron momentum p and pion momentum k):

$$\varepsilon^{(2)}_{p,\sigma} = \int \frac{d^3p' d^3k}{(2\pi)^3} \sum_{\sigma'} \frac{\left| \langle k_\pi | \langle p',\sigma' | c^+_{p',\sigma'} a_k{}^+ c_{p,\sigma} \left(\frac{1}{\sqrt{V}} \sqrt{\frac{\hbar c^2}{2\omega(k)}} \delta_{\sigma,\sigma'} \delta^3\left(\frac{\vec{p}}{\hbar} - \vec{k} - \frac{\vec{p}'}{\hbar} \right) \right) | p,\sigma\rangle |0_\pi\rangle \right|^2}{\varepsilon(p,0) - \varepsilon(p',k) + i\eta}$$

$$= \hbar c^2 \int \frac{d^3p' d^3k}{(2\pi)^3} \frac{1}{\omega(k)} \frac{\left| \langle k_\pi | \langle p',\sigma | c^+_{p',\sigma'} a_k{}^+ c_{p,\sigma} \delta^3\left(\frac{\vec{p}}{\hbar} - \vec{k} - \frac{\vec{p}'}{\hbar} \right) | p,\sigma\rangle |0_\pi\rangle \right|^2}{\varepsilon(p,0) - \varepsilon(p',k) + i\eta}$$

$$= \hbar c^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega(k)} \frac{\left| \langle k_\pi | \langle \left| \frac{\vec{p}}{\hbar} - \vec{k} \right|, \sigma | c^+_{\left|\frac{\vec{p}}{\hbar} - \vec{k}\right|,\sigma'} a_k{}^+ c_{p,\sigma} | p,\sigma\rangle |0_\pi\rangle \right|^2}{\varepsilon(p) - \varepsilon(|\vec{p} - \hbar\vec{k}|) - \varepsilon(\pi_k) + i\eta}$$

$$= \hbar c^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega(k)} \frac{1}{\varepsilon(p) - \varepsilon(|\vec{p} - \hbar k|) - \varepsilon(\pi_k) + i\eta}$$

Using the expression for the energy of free particles, (being careful of the conventions used here): $\varepsilon(p) = \frac{\hbar^2 p^2}{2m_n}, \varepsilon(\pi_k) = \sqrt{(\hbar ck)^2 + (m_\pi c^2)^2}$,

$$\varepsilon^{(2)}_{p,\sigma} = \hbar c \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2 + \frac{m_\pi{}^2}{\hbar^2}}} \frac{1}{\frac{\hbar^2 \vec{p}^2}{2m_n} - \frac{(\hbar\vec{p} - \hbar\vec{k})^2}{2m_n} - \sqrt{(\hbar c\vec{k})^2 + (m_\pi c^2)^2} + i\eta}$$

$$= \hbar c \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2 + \frac{m_\pi{}^2}{\hbar^2}}} \frac{1}{\frac{\hbar^2 \vec{p}\cdot\vec{k}}{2m_n} - \frac{\hbar^2 k^2}{2m_n} - \sqrt{(\hbar c\vec{k})^2 + (m_\pi c^2)^2} + i\eta}$$

$$= \frac{\hbar c}{(2\pi)^2} \int k^2 dk d\cos\theta \frac{1}{\sqrt{k^2 + \frac{m_\pi{}^2}{\hbar^2}}} \frac{1}{\frac{\hbar^2 pk\cos\theta}{2m_n} - \frac{\hbar^2 k^2}{2m_n} - \sqrt{(\hbar ck)^2 + (m_\pi c^2)^2} + i\eta}$$

$$= \frac{\hbar c}{(2\pi)^2} \int \frac{k^2 dk}{\sqrt{k^2 + \frac{m_\pi{}^2}{\hbar^2}}} \left(\frac{2m_n}{\hbar^2 pk} \right) \ln \left| \frac{\frac{\hbar^2 pk}{2m_n} - \frac{\hbar^2 k^2}{2m_n} - \sqrt{(\hbar ck)^2 + (m_\pi c^2)^2} + i\eta}{-\frac{\hbar^2 pk}{2m_n} - \frac{\hbar^2 k^2}{2m_n} - \sqrt{(\hbar ck)^2 + (m_\pi c^2)^2} + i\eta} \right|$$

This integral is extraordinarily difficult to perform.

c) Is there an imaginary part of the neutron energy shift in second order? Explain its origin.

It is clear from the form of the natural logarithm that this result will have an imaginary part. The real, principal part gives the energy shift of the neutron and the imaginary part of the integral gives the decay lifetime of the neutron. This is the origin of Fermi's Golden Rule.