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Quantum Mechanics 3 Homework 9

1) **Derive the dispersion relation for plasmons in the long wavelength limit. Show that it is of the form $\omega(k) = \pm\omega(0)\left[1 + \alpha k^2 + \ldots\right]$ and obtain $\omega(0)$ and α.**

Consider a free electron gas in its ground state.

Here, the density operator is $\rho_{\vec{q}} = \sum_{p,\sigma} a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}$.

While we wish to examine density fluctuations and thus the equation of motion of density, I will consider the particle-hole operator, a component $a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}$ of the density operator.

Then, the single-particle Hamiltonian is

$$H_0 = \sum_p \varepsilon(p) a^+_{\vec{p}} a_{\vec{p}} ,$$

The interaction Hamiltonian is:

$$H_I = \frac{1}{2V} \sum_k V_k \sum_{p,q,\sigma_1,\sigma_2} a^+_{\vec{p}+\vec{k},\sigma_1} a^+_{\vec{q}-\vec{k},\sigma_2} a_{\vec{q},\sigma_2} a_{\vec{p},\sigma_1}$$

Above, V_k is the Coulomb interaction.

Obtaining the equation of motion for this operator, then, I have:

$$-i\hbar \frac{d}{dt}\left(a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}\right) = \left[H_0 + H_I , a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}\right] = \left[H_0 , a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}\right] + \left[H_I , a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}\right]$$

$$\left[H_0 , a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}\right] = \left[\sum_{p'} \varepsilon(p') a^+_{\vec{p}'} a_{\vec{p}'} , a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}\right] = \sum_{p'} \varepsilon(p') \left[a^+_{\vec{p}'} a_{\vec{p}'} , a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}\right]$$
$$= \sum_{p'} \varepsilon(p') \delta_{\vec{p}',\vec{p}+\vec{q}} a^+_{\vec{p}',\sigma} a_{\vec{p},\sigma} + \sum_{p'} \varepsilon(p') \delta_{\vec{p}',\vec{q}} a_{\vec{p}',\sigma} a^+_{\vec{p}+\vec{q},\sigma} = \varepsilon(\vec{p}+\vec{q}) a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma} - \varepsilon(\vec{p}) a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}$$
$$= \left(\varepsilon(\vec{p}+\vec{q}) - \varepsilon(\vec{p})\right) a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}$$

$$\left[H_I , a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}\right] = \left[\frac{1}{2V} \sum_k V_k \sum_{p',q',\sigma_1,\sigma_2} a^+_{\vec{p}'+\vec{k},\sigma_1} a^+_{\vec{q}'-\vec{k},\sigma_2} a_{\vec{q}',\sigma_2} a_{\vec{p}',\sigma_1} , a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}\right]$$
$$= \frac{1}{2V} \sum_k V_k \left\{\left(a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p}+\vec{k},\sigma} - a^+_{\vec{p}+\vec{q}-\vec{k},\sigma} a_{\vec{p},\sigma}\right)\rho_q + \rho_q \left(a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p}+\vec{k},\sigma} - a^+_{\vec{p}+\vec{q}-\vec{k},\sigma} a_{\vec{p},\sigma}\right)\right\}$$

In order to linearize this relationship, I use the Random Phase Approximation: namely, then, with $|FS\rangle$ the Fermi sphere, that

$$a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p}+\vec{k},\sigma} \to \langle FS | a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p}+\vec{k},\sigma} | FS \rangle = n_{\vec{p}+\vec{k}} \delta_{\vec{k},\vec{q}}$$

$$a^+_{\vec{p}+\vec{q}-\vec{k},\sigma} a_{\vec{p},\sigma} \to \langle FS | a^+_{\vec{p}+\vec{q}-\vec{k},\sigma} a_{\vec{p},\sigma} | FS \rangle = n_{\vec{p}} \delta_{\vec{k},\vec{q}}$$

Now since any collective excitation in charge density has a time evolution like $e^{i\omega(k)t} e^{-\frac{\gamma(k)}{\hbar}t}$, I can apply the derivative to write:

$$-i\hbar \frac{d}{dt}\left(a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}\right) = \left[H_0 + H_I, a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}\right]$$

$$(\hbar\omega + i\gamma) a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma} = \left(\varepsilon(\vec{p}+\vec{q}) - \varepsilon(\vec{p})\right) a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma} - \frac{V_q}{V}\left(n_{p+q} - n_p\right)\rho_q$$

$$a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma} = \frac{V_q}{V} \frac{n_p - n_{p+q}}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})} \rho_k$$

Summing over all $\vec{p}$ on both sides,

$$\rho_q = \frac{2V_q}{V} \sum_{\vec{p}} \frac{n_p - n_{p+q}}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})} \rho_q$$

$$1 = \frac{2V_q}{V} \sum_{\vec{p}} \frac{n_p - n_{p+q}}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})}$$

$$1 = 2V_q \int \frac{d^3p}{(2\pi)^3} \frac{n_p - n_{p+q}}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})}$$

$$1 = 2V_q \int \frac{d^3p}{(2\pi)^3} \left[\frac{n_p}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})} - \frac{n_{p+q}}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})}\right]$$

Re-indexing,

$$1 = 2V_q \int \frac{d^3p}{(2\pi)^3} \left[\frac{n_p}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})} - \frac{n_p}{\hbar\omega + i\gamma - \varepsilon(\vec{p}) + \varepsilon(\vec{p}-\vec{q})}\right]$$

Using

$$\varepsilon(\vec{p}) \to \frac{\hbar^2 p^2}{2m} \quad \varepsilon(\vec{p}+\vec{q}) \to \frac{\hbar^2 p^2}{2m} + \frac{\hbar^2 q^2}{2m} - \frac{pq\cos\theta\hbar^2}{m}$$

where θ is the angle between $\vec{p}$ and $\vec{q}$, I may then write:

$$1 = 2V_q \int \frac{d^3p}{(2\pi)^3} n_p \frac{q^2\hbar^2}{m} \frac{1}{\left(\hbar\omega + i\gamma - \frac{pq\cos\theta\hbar^2}{m}\right)^2 - \left(\frac{q^2\hbar^2}{2m}\right)^2}$$

Consider undamped oscillations with $\gamma \to 0$. Further, take the Coulomb potential $V_q = \frac{4\pi e^2}{q^2}$. Then I get (for $q = 0$):

$$1 = \frac{4\pi e^2}{m\omega(0)^2}\left[2\int \frac{d^3p}{(2\pi)^3} n_p\right]$$

$$n \equiv \left[2\int \frac{d^3p}{(2\pi)^3} n_p\right]$$

$$\therefore \omega(0) = \omega_{pl} = \sqrt{\frac{4\pi e^2 n}{m}}$$

Now expanding in the limit of small $\frac{q}{\omega}$, I get:

$$\to 1 = 2V_0 \int \frac{d^3p}{(2\pi)^3} n_p \frac{q^2}{m\omega^2} \frac{1}{\left(1 - \frac{\hbar pq\cos\theta}{\omega m}\right)^2}$$

Now expanding I get:

$$\to 1 = 2V_q \int \frac{d^3p}{(2\pi)^3} n_p \frac{q^2}{m\omega^2}\left[1 + 2\left(\frac{\hbar pq\cos\theta}{\omega m}\right) + 3\left(\frac{\hbar pq\cos\theta}{\omega m}\right)^2 + \ldots\right]$$

$$= 2V_q \int \frac{p^2dpd(\cos\theta)}{(2\pi)^2} n_p \frac{q^2}{m\omega^2}\left[1 + 2\left(\frac{\hbar pq\cos\theta}{\omega m}\right) + 3\left(\frac{\hbar pq\cos\theta}{\omega m}\right)^2 + \ldots\right]$$

$$= 2V_q \int \frac{p^2dp}{(2\pi)^2} n_p \frac{q^2}{m\omega^2}\left[2 + 2\left(\frac{\hbar pq}{\omega m}\right)^2 + \ldots\right]$$

The final simplification was possible since the terms odd in cosine will integrate out.

Now, with $n_p = \theta(p_F - p)$, this integration gives:

$$1 = 2V_q \int_0^{p_F} \frac{p^2\,dp}{(2\pi)^2}\frac{q^2}{m\omega^2}\left[2+2\left(\frac{\hbar pq}{\omega m}\right)^2+\dots\right]$$

$$= \frac{2V_q}{2\pi^2}\frac{q^2}{m\omega^2}\left[2\frac{p_F^3}{3}+2\frac{p_F^5}{5}\left(\frac{\hbar q}{\omega m}\right)^2+\dots\right]$$

Or, with $n \equiv \left[2\int\frac{d^3p}{(2\pi)^3}n_p\right] = \frac{2}{(2\pi)^3}\frac{4}{3}\pi p_F^3$,

$$1=\frac{4\pi e^2 n}{m\omega^2}\left[1+\frac{3}{5}\left(\frac{p_F\hbar q}{\omega m}\right)^2+\dots\right]$$

Writing the ansatz $\omega(k)=\omega(0)+\alpha k^2+\dots$, and keeping only terms up to k^2 so that

$$\omega(k)=\omega(0)+\alpha k^2+\dots$$
$$\omega^2(k)=\omega(0)^2+2\alpha\omega(0)k^2+\dots$$
$$\omega^4(k)=\omega(0)^4+4\alpha\omega(0)^3k^2+\dots$$

One can obtain:
(I am not clear on the procedure by which this series is partially inverted):

$$\omega(k)=\pm\omega_{pl}\left[1+\frac{9}{10}\left(\frac{k}{k_{TF}}\right)^2+\dots\right]$$

$$k_{TF}=\sqrt{\frac{6\pi n e^2(2\pi)}{p_F^2}}$$

This is the Thomas-Fermi wave-vector.

2) **<u>Zero Sound in Neutral Fermi Gases</u>: Superfluid ^3He and neutron nuclear matter can be described well by neutral Fermi gases with a short range pair interaction with Fourier transform V_q such that $\lim_{q\to 0}V_q=V_0>0$ and finite.**

Use the RPA approximation to find the dispersion relation for density fluctuations in the long wavelength limit. Show that $\omega(\vec{k})=c|k|$ and find c both in the strong-coupling $V_0 >> \frac{\hbar^2}{mk_F}$ and weak-coupling $V_0 << \frac{\hbar^2}{mk_F}$ limits (neglect the contribution of order k^4 in the Lindhard function and propose $\omega(\vec{k})=c|k|$--check that this is consistent.) These density fluctuation excitations with dispersion relations $\omega(\vec{k})=c|k|$ are called zero sound. Argue

that the imaginary part of the Lindhard function for small damping $\gamma \to 0$ results from the "break-up" of the density fluctuations into particle-hole pairs. Compared to the case of plasmons, is there any difference?

Here the interaction Hamiltonian is:

$$H_I = \frac{1}{2V}\sum_k V_k \rho_k \rho_{-k} = \frac{1}{2V}\sum_k V_k \sum_{p,q,\sigma_1,\sigma_2} a^+_{\vec{p}+\vec{k},\sigma_1} a_{\vec{p},\sigma_1} a^+_{\vec{q}-\vec{k},\sigma_2} a_{\vec{q},\sigma_2} \ .$$

Similarly toas in problem (1): then, using the result from problem (1), I take the constraint from the limit of small $\frac{q}{\omega}$ with zero damping:

$$1 = 2V_q \frac{\hbar^2 q^2}{m}\int \frac{d^3 p}{(2\pi)^3} n_p \frac{1}{\left(\hbar\omega - \frac{\hbar^2 pq\cos\theta}{m}\right)^2}$$

Inside the Fermi sphere, I have $n_p = \theta(|p| - p_F)$. Then with $\omega(\bar{q})$, I have:

$$\int \frac{d^3 p}{(2\pi)^3} \frac{\theta(p - p_F)}{\left(\hbar\omega - \frac{\hbar^2 pq\cos\theta}{m}\right)^2}$$

$$= \int \frac{p^2 dp d(\cos\theta)}{(2\pi)^2} \frac{\theta(p - p_F)}{\left(\hbar\omega - \frac{\hbar^2 pq\cos\theta}{m}\right)^2}$$

$$= \int \frac{p^2 dp}{(2\pi)^2} \frac{2m^2\theta(p - p_F)}{\hbar^2\left(\omega^2 m^2 - \hbar^2 p^2 q^2\right)} = \frac{1}{(2\pi)^2}\frac{2m^2 p_F}{\hbar^4 q^2}\left(\frac{m\omega}{\hbar q p_F} ArcTanh\left(\frac{p_F \hbar q}{m\omega}\right) - 1\right)$$

$$= \frac{1}{(2\pi)^2}\frac{2p_F m^2}{\hbar^4 q^2}\left(\frac{1}{2}\frac{m\omega}{\hbar q p_F}\ln\left|\frac{1+\frac{p_F\hbar q}{m\omega}}{1-\frac{p_F\hbar q}{m\omega}}\right| - 1\right) = \frac{1}{(2\pi)^2}\frac{2p_F m^2}{\hbar^4 q^2}\left(\frac{1}{2}\frac{m\omega}{\hbar q p_F}\ln\left|\frac{1+\frac{m\omega}{p_F\hbar q}}{1-\frac{m\omega}{p_F\hbar q}}\right| - 1\right)$$

Where the Lindhard function is $L(x) = \frac{1}{2}x\ln\left|\frac{1+x}{1-x}\right| - 1$

Then I have (*Thanks to Chip for an explanation as to why these expansions are appropriate*):

$$1 = 2V_q \frac{\hbar^2 q^2}{m}\left(\frac{1}{(2\pi)^2}\frac{2p_F m^2}{\hbar^4 q^2}\left(\frac{1}{2}\frac{m\omega}{\hbar q p_F}\ln\left|\frac{1+\frac{m\omega}{\hbar q p_F}}{1-\frac{m\omega}{\hbar q p_F}}\right|-1\right)\right)$$

$$1 = V_q \frac{p_F m}{\pi^2\hbar^2}\left(\frac{1}{2}\frac{m\omega}{\hbar q p_F}\ln\left|\frac{1+\frac{m\omega}{\hbar q p_F}}{1-\frac{m\omega}{\hbar q p_F}}\right|-1\right)$$

However, in the limit of long wavelength and small $\vec{q}$, I would like to see all of the q-dependencies vanish in order to prevent this result from diverging. The only way to satisfy this is to have $\omega = c|\vec{q}|$, for:

$$1 = V_0 \frac{p_F m}{\pi^2\hbar^2}\left(\frac{1}{2}\frac{mc}{\hbar p_F}\ln\left|\frac{1+\frac{mc}{\hbar p_F}}{1-\frac{mc}{\hbar p_F}}\right|-1\right)$$

In the strong-coupling case, $V_0 >> \frac{\hbar^2}{mp_F}$, I have that $\frac{\hbar^2}{mp_F V_0} << 1$. In this regime, then,

$$1 >> \frac{\hbar^2}{p_F m V_0} = \frac{1}{\pi^2}\left(\frac{1}{2}\frac{mc}{\hbar p_F}\ln\left|\frac{1+\frac{mc}{\hbar p_F}}{1-\frac{mc}{\hbar p_F}}\right|-1\right)$$

Define $x \equiv \frac{mc}{\hbar p_F}$. Now I consider the limits of the RHS, $\frac{1}{\pi^2}\left(\frac{1}{2}x\ln\left|\frac{1+x}{1-x}\right|-1\right)$. If $x \to 0$, then I get

$$x \to 0 : \frac{1}{\pi^2}\left(\frac{1}{2}x\ln\left|\frac{1+x}{1-x}\right|-1\right) \approx \frac{1}{\pi^2}\left(x^2-1\right)$$

This is not particularly helpful, however, since the leftover constant term will wash-out the desired dependence on V_0. Instead, take the expansion $x \to \infty$. In this case,

$$x \to \infty : \frac{1}{\pi^2}\left(\frac{1}{2}x\ln\left|\frac{1+x}{1-x}\right|-1\right) \approx \frac{1}{\pi^2}\left(\frac{1}{3x^2}\right)$$

Then:

$$\frac{\hbar^2}{p_F m V_0} = \frac{1}{3\pi^2}\left(\frac{\hbar p_F}{mc}\right)^2$$

$$\sqrt{\frac{p_F{}^3 V_0}{3m\pi^2}} = c$$

In the weak-coupling case, $V_0 << \frac{\hbar^2}{mk_F}$ and so $\frac{\hbar^2}{mp_F V_0} >> 1$.

Then I have:

$$1 << \frac{\hbar^2}{p_F m V_0} = \frac{1}{\pi^2}\left(\frac{1}{2} x \ln\left|\frac{1+x}{1-x}\right| - 1\right)$$

If the LHS is enormous, then, I would like the logarithm to diverge in this region.

Expanding in $x \equiv \frac{mc}{\hbar p_F} \to 1+\varepsilon$, then:

$$\frac{\hbar^2}{p_F m V_0} = \frac{1}{\pi^2}\left(\frac{1}{2} x \ln\left|\frac{1+x}{1-x}\right| - 1\right)$$

$$\frac{\hbar^2}{p_F m V_0} = \frac{1}{\pi^2}\left(\frac{\ln 2}{2} - \frac{\ln(x-1)}{2} - 1\right) \to \frac{2\pi^2\hbar^2}{p_F m V_0} - 2 + \ln 2 = \ln(x-1)$$

$$\ln(x-1) = \frac{-2\pi^2\hbar^2}{p_F m V_0} - 2 + \ln 2$$

$$x - 1 = 2e^{\frac{-2\pi^2\hbar^2}{p_F m V_0} - 2}$$

$$x = 2e^{\frac{-2\pi^2\hbar^2}{p_F m V_0} - 2} + 1$$

$$c = \frac{\hbar p_F}{m}\left[2e^{\frac{-2\pi^2\hbar^2}{p_F m V_0} - 2} + 1\right]$$

With damping included, however, the formula derived in problem (1) was:

$$Lindhard = 2V_0 \int \frac{d^3 p}{(2\pi)^3} n_p \frac{q^2\hbar^2}{m} \frac{1}{\left(\hbar\omega + i\gamma - \frac{pq\cos\theta\hbar^2}{m}\right)^2 - \left(\frac{q^2\hbar^2}{2m}\right)^2}.$$

The imaginary portion of this (from the principal value definition) is:

$$\mathrm{Im}\left(\frac{1}{x - i\eta}\right) \to \pi\delta(x)$$

Here I make a bit of a leap: that since γ is assumed small, then I can pull a linear part out of the square and it remains small, and the real contribution is square in γ and is extraordinarily small. Then:

$$\mathrm{Im}(Lindhard) = 2\pi V_0 \int \frac{d^3 p}{(2\pi)^3} n_p \frac{q^2\hbar^2}{m} \delta\left(\left(\hbar\omega - \frac{pq\cos\theta\hbar^2}{m}\right)^2 - \left(\frac{q^2\hbar^2}{2m}\right)^2\right)$$

And I see that it arises from the particle-hole pair breakup since $\varepsilon(\vec{p}+\vec{q})-\varepsilon(\vec{p})$ is the energy of a particle-hole pair!

3) **<u>Landau Damping</u>: Compute the imaginary part of the Lindhard function in the small damping limit $\gamma \to 0$.**

a) **After a suitable change of variable, show that $[L..F.]$, the Lindhard Function, has**

$$\mathrm{Im}[L.F.] = -2\pi \int \frac{d^3p}{(2\pi)^3} n_p \left[\delta\left(\hbar\omega - \varepsilon(k) + \hbar^2 \frac{\vec{p}\cdot\vec{k}}{m} \right) - \delta\left(\hbar\omega + \varepsilon(k) - \hbar^2 \frac{\vec{p}\cdot\vec{k}}{m} \right) \right].$$

Namely, the imaginary part is an odd function of frequency.

Taking a result from problem (1), the Lindhard function is given by the integral on the right-hand side (except for V_q) of:

$$1 = 2V_q \int \frac{d^3p}{(2\pi)^3} \left[\frac{n_p}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})} - \frac{n_p}{\hbar\omega + i\gamma - \varepsilon(\vec{p}) + \varepsilon(\vec{p}-\vec{q})} \right]$$

In the limit of small γ, then, this becomes a principal value integral: namely,

$$\frac{1}{x - i\eta} = P\left(\frac{1}{x}\right) + i\pi\delta(x) \quad e.g. \quad \frac{1}{x + i\eta} = P\left(\frac{1}{x}\right) - i\pi\delta(x)$$

Taking the imaginary part corresponding to the second term, then, I have:

$$2V_q \int \frac{d^3p}{(2\pi)^3} n_p \left[\frac{1}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})} - \frac{1}{\hbar\omega + i\gamma - \varepsilon(\vec{p}) + \varepsilon(\vec{p}-\vec{q})} \right]$$

$$\to -2\pi i V_q \int \frac{d^3p}{(2\pi)^3} n_p \left[\delta(\hbar\omega - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})) - \delta(\hbar\omega - \varepsilon(\vec{p}) + \varepsilon(\vec{p}-\vec{q})) \right]$$

Also recalling:

$$\varepsilon(\vec{p}) \to \frac{\hbar^2 p^2}{2m} \quad \varepsilon(\vec{p}+\vec{q}) \to \frac{\hbar^2 p^2}{2m} + \frac{\hbar^2 q^2}{2m} + \frac{\vec{p}\cdot\vec{q}\hbar^2}{m}$$

Then I have:

$$1 = -2\pi i V_q \int \frac{d^3p}{(2\pi)^3} n_p \left[\delta\left(\hbar\omega - \frac{\hbar^2 q^2}{2m} + \frac{\vec{p}\cdot\vec{q}\hbar^2}{m} \right) - \delta\left(\hbar\omega + \frac{\hbar^2 q^2}{2m} - \frac{\vec{p}\cdot\vec{q}\hbar^2}{m} \right) \right]$$

b) Write $\int d^3 p n_p = 2\pi \int_0^{p_F} dp p^2 \int_{-1}^{1} d(\cos\theta)$ and $\vec{p}\cdot\vec{k} = pk\cos\theta$. Use the δ function to integrate in $\cos\theta$, this gives a constraint on the values of p as a function of k,ω. Finally integrate in p being careful with these constraints.

$$= -\frac{iV_q}{2\pi}\int_0^{p_F} dp p^2 \int_{-1}^{1} d\cos\theta \left[\delta\left(\hbar\omega - \frac{\hbar^2 q^2}{2m} + \frac{pq\hbar^2\cos\theta}{m}\right) - \delta\left(\hbar\omega + \frac{\hbar^2 q^2}{2m} - \frac{pq\hbar^2\cos\theta}{m}\right)\right]$$

Note well that: $\int_{-1}^{1}\delta(A+Bx)dx = \frac{1}{|B|}\left[\theta(-A-B)\theta(A-B)\theta(-B) + \theta(B)\theta(-A+B)\theta(A+B)\right]$

Then, omitting irrelevant parts since $\hbar, p, q, m > 0$:

$$= -\frac{iV_q}{2\pi}\int_0^{p_F} dp p^2 \frac{m}{pq\hbar^2}\left[\begin{array}{l}\theta\left(-\hbar\omega + \frac{\hbar^2 q^2}{2m} + \frac{pq\hbar^2}{m}\right)\theta\left(\hbar\omega - \frac{\hbar^2 q^2}{2m} + \frac{pq\hbar^2}{m}\right) \\ -\theta\left(-\hbar\omega - \frac{\hbar^2 q^2}{2m} + \frac{pq\hbar^2}{m}\right)\theta\left(\hbar\omega + \frac{\hbar^2 q^2}{2m} + \frac{pq\hbar^2}{m}\right)\end{array}\right]$$

Examining the constraints imposed by these step functions, I see that for the top one to take effect:

$$p > \frac{m\omega}{\hbar q} - \frac{q}{2} \quad \textit{and} \quad p > -\frac{m\omega}{\hbar q} + \frac{q}{2} \rightarrow p > \left|\frac{m\omega}{\hbar q} - \frac{q}{2}\right|$$

From the bottom one, I see that for the bottom one to take effect:

$$p > \frac{m\omega}{\hbar q} + \frac{q}{2} \quad \textit{and} \quad p > -\frac{m\omega}{\hbar q} - \frac{q}{2} \rightarrow p > \left|\frac{m\omega}{\hbar q} + \frac{q}{2}\right|$$

So I see that my integral vanishes except where $\left|\frac{m\omega}{\hbar q} - \frac{q}{2}\right| < p < \left|\frac{m\omega}{\hbar q} + \frac{q}{2}\right|$ if $\omega > 0$ and where $\left|\frac{m\omega}{\hbar q} + \frac{q}{2}\right| < p < \left|\frac{m\omega}{\hbar q} - \frac{q}{2}\right|$ if $\omega < 0$.

Now I have, performing the very simple integral that remains,

$$=-\frac{imV_q}{4\pi q\hbar^2}\left[\left[p^2\right]_{p=\left|\frac{m\omega}{\hbar q}-\frac{q}{2}\right|}^{p=\min\left(p_F,\left|\frac{m\omega}{\hbar q}+\frac{q}{2}\right|\right)}\theta(w)+\left[p^2\right]_{p=\left|\frac{m\omega}{\hbar q}+\frac{q}{2}\right|}^{p=\min\left(p_F,\left|\frac{m\omega}{\hbar q}-\frac{q}{2}\right|\right)}\theta(-\omega)\right]$$

$$=-\frac{imV_q}{4\pi q\hbar^2}\left[\begin{array}{l}\left[\left(\min\left(p_F,\left|\frac{m\omega}{\hbar q}+\frac{q}{2}\right|\right)\right)^2-\left(\frac{m\omega}{\hbar q}-\frac{q}{2}\right)^2\right]\theta(w)\\ +\left[\left(\min\left(p_F,\left|\frac{m\omega}{\hbar q}-\frac{q}{2}\right|\right)\right)^2-\left(\frac{m\omega}{\hbar q}+\frac{q}{2}\right)^2\right]\theta(-w)\end{array}\right]$$

4) **The dielectric function in the Random Phase Approximation is given by** $\varepsilon(\vec{k},\omega)=1-2V_k\int\frac{d^3p}{(2\pi)^3}\frac{n_p-n_{p+k}}{\hbar\omega-\varepsilon(\vec{p}+\vec{k})+\varepsilon(\vec{p})}$**. Compute** $\varepsilon(\vec{k},\omega=0)$**. This function describes screening.**

$$\varepsilon(\vec{k},\omega=0)=1-2V_k\int\frac{d^3p}{(2\pi)^3}\frac{n_p-n_{p+k}}{\varepsilon(\vec{p})-\varepsilon(\vec{p}+\vec{k})}$$

$$=1-2V_k\int\frac{d^3p}{(2\pi)^3}\left[\frac{n_p}{\varepsilon(\vec{p})-\varepsilon(\vec{p}+\vec{k})}-\frac{n_{p+k}}{\varepsilon(\vec{p})-\varepsilon(\vec{p}+\vec{k})}\right]$$

$$re-index: p+k\to p \quad in \quad last \quad term$$

$$=1-2V_k\int\frac{d^3p}{(2\pi)^3}\left[\frac{n_p}{\varepsilon(\vec{p})-\varepsilon(\vec{p}+\vec{k})}-\frac{n_p}{\varepsilon(\vec{p}-\vec{k})-\varepsilon(\vec{p})}\right]$$

$$Note:\varepsilon(\vec{p})=\frac{\hbar^2p^2}{2m}\quad \varepsilon(\vec{p}-\vec{k})-\varepsilon(\vec{p})=\frac{\hbar^2k^2}{2m}-\frac{\hbar^2pk\cos\theta}{m}\quad \varepsilon(\vec{p})-\varepsilon(\vec{p}+\vec{k})=-\frac{\hbar^2k^2}{2m}-\frac{\hbar^2pk\cos\theta}{m}$$

However, since I may choose to integrate over $\theta\to\pi+\theta$ in the second term, these are the same and I have:

$$=1-4V_k\int\frac{d^3p}{(2\pi)^3}\frac{n_p}{\varepsilon(\vec{p})-\varepsilon(\vec{p}+\vec{k})}=1+4V_k\int\frac{d^3p}{(2\pi)^3}\frac{n_p}{\varepsilon(\vec{p}+\vec{k})-\varepsilon(\vec{p})}$$

Let me take the integral only: each integration (which Martin and Rothen, page 156, describes as “elementary”) comes from Mathematica.

$$\int \frac{d^3 p}{(2\pi)^3} \frac{n_p}{\varepsilon\left(\vec{p}+\vec{k}\right)-\varepsilon\left(\vec{p}\right)} = \int \frac{p^2 dp d(\cos\theta)}{(2\pi)^2} \frac{\theta(p-p_F)}{\dfrac{\hbar^2 k^2}{2m} - \dfrac{\hbar^2 pk\cos\theta}{m}}$$

$$= \frac{m}{4\pi^2\hbar^2 k} \int dp \frac{p}{2} \ln\left|\frac{2p+k}{2p-k}\right| \theta(p-p_F)$$

$$= \frac{m}{4\pi^2\hbar^2 k} \left[\frac{kp}{2} + \frac{1}{8}k^2 \ln|2p-k| - \frac{1}{8}k^2 \ln|2p+k| + \frac{1}{2}p^2 \ln\left|\frac{2p+k}{2p-k}\right| \right]\Bigg|_{p=0}^{p_F}$$

$$= \frac{m}{4\pi^2\hbar^2 k^2 p_F} \left[\frac{1}{2} - \frac{1}{8}\frac{k}{p_F} \ln\left|\frac{2p_F+k}{2p_F-k}\right| + \frac{1}{2}\frac{p_F}{k} \ln\left|\frac{2p_F+k}{2p_F-k}\right| \right]$$

$$= \frac{m}{4\pi^2\hbar^2 k^2 p_F} \left[\frac{1}{2} + \frac{p_F}{2k}\left(1-\left(\frac{k}{2p_F}\right)^2\right) \ln\left|\frac{2p_F+k}{2p_F-k}\right| \right]$$

Martin and Rothen, page 156, describes this integral as “elementary”. Substituting this into the original form, then,

$$\varepsilon\left(\vec{k},\omega=0\right) = 1 - \frac{mV_k}{\pi^2\hbar^2 k^2 p_F} \left[\frac{1}{2} + \frac{p_F}{2k}\left(1-\left(\frac{k}{2p_F}\right)^2\right) \ln\left|\frac{2p_F+k}{2p_F-k}\right| \right]$$