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Quantum Mechanics 3 Homework 6

1) **The classical Hamiltonian of a particle with charge e in the presence of electromagnetic fields is $H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right)^2 + e\phi(\vec{x},t)$ with $\vec{x}$ being the coordinate of the particle. From Hamilton's equations obtain the equation of motion for the particle and confirm the Lorentz force.**

$$Note: \vec{\nabla}\left[\vec{B}\cdot\vec{B}\right] = 2\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right) + 2\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}$$

Then, noticing that $\vec{\nabla}_{\vec{p}} \times \left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right) = \vec{\nabla}_{\vec{p}} \times \vec{p} = 0$, I have:

$$\vec{\nabla}_{\vec{p}} H = \vec{\nabla}_{\vec{p}}\left[\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right)^2 + e\phi(\vec{x},t)\right] = \dot{\vec{x}}$$

$$\dot{\vec{x}} = \frac{1}{m}\left[\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right)\cdot\vec{\nabla}_{\vec{p}}\right]\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right) = \frac{1}{m}\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right)$$

$$\vec{p} = m\dot{\vec{x}} + \frac{e}{c}\vec{A}(\vec{x},t)$$

$$\dot{\vec{p}} = \frac{d}{dt}\vec{p} = \left[\frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right]\vec{p} = m\ddot{\vec{x}} + \frac{e}{c}\dot{\vec{x}}\cdot\vec{\nabla}\vec{A}(\vec{x},t) + \frac{e}{c}\frac{\partial\vec{A}(\vec{x},t)}{\partial t}$$

Further,

$$\vec{\nabla}_{\vec{x}} H = -\dot{\vec{p}}$$

$$-\dot{\vec{p}} = \frac{1}{2m}\left[2\left(\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right)\cdot\vec{\nabla}_{\vec{x}}\right)\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right) + 2\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right)\times\left(\vec{\nabla}\times\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right)\right)\right] + e\vec{\nabla}_{\vec{x}}\phi(\vec{x},t)$$

$$Substituting:$$

$$-\dot{\vec{p}} = \frac{1}{2m}\left[2\left(m\dot{\vec{x}}\cdot\vec{\nabla}_{\vec{x}}\right)\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right) + 2m\dot{\vec{x}}\times\left(\vec{\nabla}_{\vec{x}}\times\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right)\right)\right] + e\vec{\nabla}_{\vec{x}}\phi(\vec{x},t)$$

$$Note: \vec{\nabla}_{\vec{x}}\vec{p} = 0 \quad \vec{\nabla}_{\vec{x}}\times\vec{p} = 0$$

$$-\dot{\vec{p}} = -m\ddot{\vec{x}} - \frac{e}{c}\dot{\vec{x}}\cdot\vec{\nabla}\vec{A}(\vec{x},t) - \frac{e}{c}\frac{\partial\vec{A}(\vec{x},t)}{\partial t} = -\frac{e}{c}\left[\left(\dot{\vec{x}}\cdot\vec{\nabla}_{\vec{x}}\right)\vec{A}(\vec{x},t) + \dot{\vec{x}}\times\left(\vec{\nabla}_{\vec{x}}\times\vec{A}(\vec{x},t)\right)\right] + e\vec{\nabla}_{\vec{x}}\phi(\vec{x},t)$$

$$Use: \vec{B}(\vec{x},t) \equiv \vec{\nabla}_{\vec{x}}\times\vec{A}(\vec{x},t) \quad \vec{E}(\vec{x},t) \equiv -\vec{\nabla}_{\vec{x}}\phi(\vec{x},t) - \frac{1}{c}\frac{\partial\vec{A}(\vec{x},t)}{\partial t}$$

$$m\ddot{\vec{x}} = \frac{e}{c}\left[\dot{\vec{x}}\times\vec{B}(\vec{x},t)\right] + e\vec{E}(\vec{x},t)$$

Since $\vec{F} = m\ddot{x}$, this is the Lorentz force (in CGS units).

2) **Obtain an expression for the correlation function for a free gas (bosons and fermions)** $\left\langle \psi^+(\vec{x})\psi(\vec{y})\right\rangle = \dfrac{Tr\left[e^{-\beta(\hat{H}-\mu\hat{N})}\psi^+(\vec{x})\psi(\vec{y})\right]}{Tr\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right]}$

a) **Leave it expressed in terms of integrals.**

For free particles,

$$\left\langle \psi^+(\vec{x})\psi(\vec{y})\right\rangle = \frac{1}{V}\sum_{\vec{p}_1,\sigma_1}\sum_{\vec{p}_2,\sigma_2} e^{\frac{i}{\hbar}\vec{p}_1\cdot\vec{x}} e^{-\frac{i}{\hbar}\vec{p}_2\cdot\vec{y}} \left\langle a^+_{\vec{p}_1,\sigma_1} a_{\vec{p}_2,\sigma_2}\right\rangle = \frac{1}{V}\sum_{\vec{p},\sigma} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{y}} \left\langle n_{\vec{p},\sigma}\right\rangle$$

In the final step above, I have noticed that the Fock-state value of this operator must be zero except where $\vec{p}_1 = \vec{p}_2, \sigma_1 = \sigma_2$ and identified the result as the number operator. It is well-known that the distribution function for a free gas is:

$$\mp \begin{matrix} Bose \\ Fermi \end{matrix}$$

$$\left\langle \hat{N}\right\rangle = \sum_{\vec{p},\sigma} n_{\vec{p},\sigma} = \sum_{\vec{p},\sigma} \frac{1}{e^{\beta(\varepsilon(\vec{p})-\mu)} \mp 1}$$

$$\left\langle \psi^+(\vec{x})\psi(\vec{y})\right\rangle = \frac{1}{V}\sum_{\vec{p},\sigma} e^{\frac{i}{\hbar}\vec{p}\cdot(\vec{x}-\vec{y})} \cdot \frac{1}{e^{\beta(\varepsilon(\vec{p})-\mu)} \mp 1} = \sum_{\sigma}\int \frac{d^3p}{(2\pi)^3} e^{\frac{i}{\hbar}\vec{p}\cdot(\vec{x}-\vec{y})} \cdot \frac{1}{e^{\beta(\varepsilon(\vec{p})-\mu)} \mp 1}$$

$$Define \quad \vec{R} \equiv \frac{\vec{x}-\vec{y}}{\hbar}$$

$$Define \quad s \equiv number \quad of \quad allowed \quad values \quad of \quad \sigma$$

$$\therefore \left\langle \psi^+(\vec{x})\psi(\vec{y})\right\rangle = (2s+1)\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{R}} \cdot \frac{1}{e^{\beta(\varepsilon(\vec{p})-\mu)} \mp 1} = (2s+1)\int \frac{d^3p}{(2\pi)^3} e^{ipR\cos\theta} \cdot \frac{1}{e^{\beta(\varepsilon(\vec{p})-\mu)} \mp 1}$$

$$If \quad \varepsilon(\vec{p}) \quad spherically \quad symmetric:$$

$$= \frac{2s+1}{2\pi^2}\int p^2 dp \frac{\sin(pR)}{pR} \cdot \frac{1}{e^{\beta(\varepsilon(\vec{p})-\mu)} \mp 1} = \frac{2s+1}{2\pi^2 R}\int dp \frac{p\sin(pR)}{e^{\beta(\varepsilon(\vec{p})-\mu)} \mp 1}$$

Just as in the notes. Here, $\varepsilon(\vec{p})$ is some function taking momentum to energy; for free particles, $\varepsilon(\vec{p}) = \dfrac{p^2}{2m}$.

b) **Take the zero-temperature limit and compute** $\left\langle \psi^+(\vec{x})\psi(\vec{y})\right\rangle$ **both for Fermions and Bosons.**

For Bosons, all particles will be in the lowest state, $\vec{p}=0$. Now simply taking

$$\left\langle \psi^{+}(\vec{x})\psi(\vec{y})\right\rangle = \frac{1}{V}\sum_{\vec{p},\sigma} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{y}} \left\langle n_{\vec{p},\sigma}\right\rangle \underset{\vec{p}\to\vec{0}}{\rightarrow} \frac{1}{V}\sum_{\vec{p},\sigma}\left\langle n_{\vec{p},\sigma}\right\rangle = \frac{N}{V}.$$

For Fermions, I understand that each state is totally filled up to the Fermi level p_F. Further, up to this level the distribution $\frac{1}{e^{\beta(\varepsilon(\vec{p})-\mu)}+1}$ should be flat; namely, it should be equal to a constant $\frac{1}{e^{\beta(\varepsilon(\vec{p})-\mu)}+1} = \begin{cases} \frac{n}{(2s+1)\left(\frac{4}{3}\pi p_F{}^{3}\right)} & 0<p<p_F \\ 0 & otherwise \end{cases}$

So:

$$\frac{2s+1}{2\pi^2 R}\left[\frac{n}{(2s+1)\left(\frac{4}{3}\pi p_F^3\right)}\right]\int_0^{p_F} dpp\sin(pR)$$

$$=\frac{3n}{8\pi^3 R p_F^3}\left[\frac{-Rp_F\cos(Rp_F)+\sin(Rp_F)}{R^2}\right]$$

$$=\frac{3n}{8\pi^3}\left[\frac{-(Rp_F)\cos(Rp_F)+\sin(Rp_F)}{(p_F R)^3}\right]$$

3) **Consider the Hamiltonian for electrons and photons in the presence of an external static charge.**

a) **Obtain the matrix elements of the interaction terms for 1 and 2 photon processes in terms of the single particle eigenfunctions in the presence of the external static potential.**

$$\hat{H} = \int d^3x\psi^{+}\left[-\frac{\hbar^2\vec{\nabla}^2}{2m}+e\phi\right]\psi + \int d^3x\left[\vec{A}\cdot\psi^{+}\frac{i\hbar e}{mc}\vec{\nabla}\psi + \psi^{+}\frac{e^2}{c^2}\frac{\vec{A}^2}{2m}\psi\right]$$

Further, removing the time-dependent portion since the total Hamiltonian is not dependent on time, I have:

$$\vec{A}(\vec{x}) = \frac{c}{\sqrt{V}}\sum_k\sum_{\lambda=1}^{2}\vec{\varepsilon}_\lambda(\vec{k})\sqrt{\frac{\hbar}{2\omega(k)}}\left[a_{k,\lambda}e^{i\vec{k}\cdot\vec{x}} + a_{k,\lambda}^{+}e^{-i\vec{k}\cdot\vec{x}}\right]$$

Here $a_{k,\lambda}, a_{k,\lambda}^{+}$ are destruction and creation operators for photons, respectively.

Let $\psi = \sum_n c_n u_n(\vec{x})$, where $\left(-\frac{\hbar^2\vec{\nabla}^2}{2m}+e\phi\right)u_n(\vec{x}) = \varepsilon_n u_n(\vec{x})$,

and c_n is a possibly complex constant.

$$\hat{H}_I = \int d^3x \left[\vec{A} \cdot \psi^+ \frac{i\hbar e}{mc} \vec{\nabla} \psi + \psi^+ \frac{e^2}{c^2} \frac{\vec{A}^2}{2m} \psi \right]$$

Now the first-order terms are:

$$\hat{H}_{I,\gamma} = \int d^3x \left[\vec{A} \cdot \psi^+ \frac{i\hbar e}{mc} \vec{\nabla} \psi \right]$$

$$= \frac{c}{\sqrt{V}} \sum_k \sum_{\lambda=1}^{2} \sum_{n,n'} \int d^3x \left[\sqrt{\frac{\hbar}{2\omega(k)}} c_n^+ u_n^*(\vec{x}) \frac{i\hbar e}{mc} c_n \left(\vec{\varepsilon}_\lambda(k) \cdot \vec{\nabla} u_{n'}(\vec{x}) \right) \right] \left[a_{k,\lambda} e^{i\vec{k}\cdot\vec{x}} + a_{k,\lambda}^+ e^{-i\vec{k}\cdot\vec{x}} \right]$$

$$Define : V_{k,\lambda,n,n'} \equiv \frac{c}{\sqrt{V}} \sqrt{\frac{\hbar}{2\omega(k)}} \frac{i\hbar e}{mc} \int d^3x \left[e^{i\vec{k}\cdot\vec{x}} u_n^*(\vec{x}) \left(\vec{\varepsilon}_\lambda(k) \cdot \vec{\nabla} u_{n'}(\vec{x}) \right) \right]$$

$$= i \sqrt{\frac{\hbar^3 e^2}{2\omega(k) m^2 V}} \int d^3x \left[e^{i\vec{k}\cdot\vec{x}} u_n^*(\vec{x}) \left(\vec{\varepsilon}_\lambda(k) \cdot \vec{\nabla} u_{n'}(\vec{x}) \right) \right]$$

Similarly,

$$\tilde{V}_{k,\lambda,n,n'} \equiv i \sqrt{\frac{\hbar^3 e^2}{2\omega(k) m^2 V}} \int d^3x \left[e^{-i\vec{k}\cdot\vec{x}} u_n^*(\vec{x}) \left(\vec{\varepsilon}_\lambda(k) \cdot \vec{\nabla} u_{n'}(\vec{x}) \right) \right]$$

$$\hat{H}_{I,\gamma} = \sum_k \sum_{\lambda=1}^{2} \sum_{n,n'} c_n^+ c_n \left[V_{k,\lambda,n,n'} a_{k,\lambda} + \tilde{V}_{k,\lambda,n,n'} a_{k,\lambda}^+ \right]$$

The second-order terms are then:

$$\hat{H}_{I,\gamma\gamma} = \int d^3x \left[\psi^+ \frac{e^2}{c^2} \frac{\vec{A}^2}{2m} \psi \right]$$

$$= \frac{c^2 e^2 \hbar^2}{Vc^2 4m} \sum_{k,k'} \sum_{\lambda,\lambda'=1}^{2} \sum_{n,n'} \frac{\vec{\varepsilon}_\lambda(k) \cdot \vec{\varepsilon}_{\lambda'}(k')}{\sqrt{\omega(k)\omega(k')}} c_n^+ c_{n'} \int d^3x u_n^*(\vec{x}) \begin{pmatrix} \left[a_{k,\lambda} e^{i\vec{k}\cdot\vec{x}} + a_{k,\lambda}^+ e^{-i\vec{k}\cdot\vec{x}} \right] \\ \cdot \left[a_{k',\lambda'} e^{i\vec{k}'\cdot\vec{x}} + a_{k',\lambda'}^+ e^{-i\vec{k}'\cdot\vec{x}} \right] \end{pmatrix} u_{n'}(\vec{x})$$

Then :

$$V_{k,k',\lambda,\lambda',n,n'}^{--} = \frac{e^2\hbar^2}{V4m} \frac{\vec{\varepsilon}_\lambda(k) \cdot \vec{\varepsilon}_{\lambda'}(k')}{\sqrt{\omega(k)\omega(k')}} \int d^3x u_n^*(\vec{x}) e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} u_{n'}(\vec{x})$$

$$V_{k,k',\lambda,\lambda',n,n'}^{-+} = \frac{e^2\hbar^2}{V4m} \frac{\vec{\varepsilon}_\lambda(k) \cdot \vec{\varepsilon}_{\lambda'}(k')}{\sqrt{\omega(k)\omega(k')}} \int d^3x u_n^*(\vec{x}) e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} u_{n'}(\vec{x})$$

$$V_{k,k',\lambda,\lambda',n,n'}^{+-} = \frac{e^2\hbar^2}{V4m} \frac{\vec{\varepsilon}_\lambda(k) \cdot \vec{\varepsilon}_{\lambda'}(k')}{\sqrt{\omega(k)\omega(k')}} \int d^3x u_n^*(\vec{x}) e^{i(-\vec{k}+\vec{k}')\cdot\vec{x}} u_{n'}(\vec{x})$$

$$V_{k,k',\lambda,\lambda',n,n'}^{++} = \frac{e^2\hbar^2}{V4m} \frac{\vec{\varepsilon}_\lambda(k) \cdot \vec{\varepsilon}_{\lambda'}(k')}{\sqrt{\omega(k)\omega(k')}} \int d^3x u_n^*(\vec{x}) e^{i(-\vec{k}-\vec{k}')\cdot\vec{x}} u_{n'}(\vec{x})$$

$$\hat{H}_{I,\gamma\gamma} = \sum_{k,k'} \sum_{\lambda,\lambda'=1}^{2} \sum_{n,n'} c_n^+ c_{n'} \begin{bmatrix} a_{k,\lambda} a_{k',\lambda'} V_{k,k',\lambda,\lambda',n,n'}^{--} \\ + a_{k,\lambda} a_{k',\lambda'}^+ V_{k,k',\lambda,\lambda',n,n'}^{-+} \\ + a_{k,\lambda}^+ a_{k',\lambda'} V_{k,k',\lambda,\lambda',n,n'}^{+-} \\ + a_{k,\lambda}^+ a_{k',\lambda'}^+ V_{k,k',\lambda,\lambda',n,n'}^{++} \end{bmatrix}$$

b) Repeat the calculation with free electrons.

For free electrons, $\phi = 0$. In the previous problem, the eigenfunctions were defined by $\left(-\frac{\hbar^2 \vec{\nabla}^2}{2m} + e\phi \right) u_n(\vec{x}) = \varepsilon_n u_n(\vec{x})$, but now a continuum of electron momenta are available.

Based on this, what had been my matrix elements are now defined over a continuum. Namely, the allowed solutions of $u_n(\vec{x})$ become $u_{\vec{p}}(\vec{x}) = e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$ and my matrix elements become:

$$V_{k,\lambda}(\vec{p},\vec{p}')=i\sqrt{\frac{\hbar^3 e^2}{2\omega(\vec{k})m^2V}}e^{-i\omega(k)t}\int d^3x\left[e^{i\vec{k}\cdot\vec{x}}u^*_{\vec{p}}(\vec{x})\left(\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\nabla}u_{\vec{p}'}(\vec{x})\right)\right]$$

$$\tilde{V}_{k,\lambda}(\vec{p},\vec{p}')=i\sqrt{\frac{\hbar^3 e^2}{2\omega(\vec{k})m^2V}}e^{i\omega(k)t}\int d^3x\left[e^{-i\vec{k}\cdot\vec{x}}u^*_{\vec{p}}(\vec{x})\left(\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\nabla}u_{\vec{p}'}(\vec{x})\right)\right]$$

$$\hat{H}_{I,\gamma}=\frac{1}{(2\pi)^6}\int d^3p\,d^3p'\sum_{k,k'}\sum_{\lambda=1}^{2}c^+_p c_p\left[V_{k,\lambda}(\vec{p},\vec{p}')a_{k,\lambda}e^{-i\omega(k)t}+\tilde{V}_{k,\lambda,n,n'}(\vec{p},\vec{p}')a^+_{k,\lambda}e^{i\omega(k)t}\right]$$

$$V^{--}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}')=\frac{e^2\hbar^2}{V4m}\frac{\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k}')}{\sqrt{\omega(k)\omega(k')}}\int d^3x u^*_{\vec{p}}(\vec{x})e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}u_{\vec{p}'}(\vec{x})$$

$$\int d^3x e^{-\frac{i}{\hbar}\vec{p}'\cdot x}(\vec{x})e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}e^{\frac{i}{\hbar}\vec{p}\cdot x}=(2\pi)^3\delta^3\left(\vec{k}+\vec{k}'+\frac{\vec{p}}{\hbar}-\frac{\vec{p}'}{\hbar}\right)$$

$$\therefore V^{--}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}')=\frac{e^2\hbar^2}{V4m}\frac{\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k}')}{\sqrt{\omega(\vec{k})\omega(\vec{k}')}}\delta^3\left(\vec{k}+\vec{k}'+\frac{\vec{p}}{\hbar}-\frac{\vec{p}'}{\hbar}\right)$$

Similarly,

$$V^{-+}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}')=\frac{e^2\hbar^2}{V4m}\frac{\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k}')}{\sqrt{\omega(\vec{k})\omega(\vec{k}')}}\int d^3x u^*_{\vec{p}}(\vec{x})e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}u_{\vec{p}'}(\vec{x})=\frac{(2\pi)^3e^2\hbar^2}{V4m}\frac{\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k}')}{\sqrt{\omega(\vec{k})\omega(\vec{k}')}}\delta^3\left(\vec{k}-\vec{k}'+\frac{\vec{p}}{\hbar}-\frac{\vec{p}'}{\hbar}\right)$$

$$V^{+-}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}')=\frac{e^2\hbar^2}{V4m}\frac{\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k}')}{\sqrt{\omega(\vec{k})\omega(\vec{k}')}}\int d^3x u^*_{\vec{p}}(\vec{x})e^{i(-\vec{k}+\vec{k}')\cdot\vec{x}}u_{\vec{p}'}(\vec{x})=\frac{(2\pi)^3e^2\hbar^2}{V4m}\frac{\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k}')}{\sqrt{\omega(\vec{k})\omega(\vec{k}')}}\delta^3\left(-\vec{k}+\vec{k}'+\frac{\vec{p}}{\hbar}-\frac{\vec{p}'}{\hbar}\right)$$

$$V^{++}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}')=\frac{e^2\hbar^2}{V4m}\frac{\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k}')}{\sqrt{\omega(\vec{k})\omega(\vec{k}')}}\int d^3x u^*_{\vec{p}}(\vec{x})e^{i(-\vec{k}-\vec{k}')\cdot\vec{x}}u_{\vec{p}'}(\vec{x})=\frac{(2\pi)^3e^2\hbar^2}{V4m}\frac{\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k}')}{\sqrt{\omega(\vec{k})\omega(\vec{k}')}}\delta^3\left(-\vec{k}-\vec{k}'+\frac{\vec{p}}{\hbar}-\frac{\vec{p}'}{\hbar}\right)$$

$$\hat{H}_{I,\gamma\gamma}=\frac{1}{(2\pi)^6}\int d^3p\,d^3p'\sum_{k,k'}\sum_{\lambda,\lambda'=1}^{2}c^+_p c_p\begin{bmatrix}a_{k,\lambda}a_{k',\lambda'}V^{--}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}')\\+a_{k,\lambda}a^+_{k',\lambda'}V^{-+}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}')\\+a^+_{k,\lambda}a_{k',\lambda'}V^{+-}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}')\\+a^+_{k,\lambda}a^+_{k',\lambda'}V^{++}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}')\end{bmatrix}$$

Briefly, then, I see that here the combination $c^+_p c_p$ plays the role of a density of electrons at this energy and the matrix operators characterize the degree of energetic interaction between electrons at these levels.

4) **The first order shift in energy of a quantum state $|n\rangle$ for a perturbation described by an interaction Hamiltonian H_I is $\Delta E_n=\langle n|H_I|n\rangle$. Consider the interaction Hamiltonian between electrons and photons with no external potential $(\phi=0)$. If $|n\rangle=|1_{p_\sigma},0\rangle$ is the state of one electron of momentum p and spin component σ and the photon vacuum, find the first order energy shift. Is your answer finite? Explain.**

First, consider

$$\Delta E_n = \left\langle 1_{p_\sigma},0 \left| H_I \right| 1_{p_\sigma},0 \right\rangle .$$

I see that the only valid term from the interaction Hamiltonian is one that creates and then destroys the same photon. Namely,

$$V^{-+}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}') = \frac{(2\pi)^3 e^2 \hbar^2}{V4m} \frac{\vec{\varepsilon}_\lambda(\vec{k}) \cdot \vec{\varepsilon}_{\lambda'}(\vec{k}')}{\sqrt{\omega(\vec{k})\omega(\vec{k}')}} \delta^3\left(\vec{k} - \vec{k}' + \frac{\vec{p}}{\hbar} - \frac{\vec{p}'}{\hbar}\right)$$

and

$$\hat{H}_{I,\gamma\gamma} = \frac{1}{(2\pi)^6} \int d^3p d^3p' \sum_{k,k'} \sum_{\lambda,\lambda'=1}^{2} c_p^+ c_p \left[a_{k,\lambda} a^+_{k',\lambda'} V^{-+}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}')\right]$$

Simplifying, then, I have:

$$\left\langle 1_{p_\sigma},0 \left| \hat{H}_{I,\gamma\gamma} \right| 1_{p_\sigma},0 \right\rangle$$

$$= \left\langle 1_{p_\sigma},0 \right| \frac{1}{(2\pi)^6} \int d^3p d^3p' \sum_{k,k'} \sum_{\lambda,\lambda'=1}^{2} c_p^+ c_p \left[a_{k,\lambda} a^+_{k',\lambda'} V^{-+}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}')\right] \left| 1_{p_\sigma},0 \right\rangle$$

$$= \frac{1}{(2\pi)^6} \int d^3p d^3p' \delta^3(\vec{p} - \vec{p}_\sigma) \sum_{k,k'} \sum_{\lambda,\lambda'=1}^{2} \left[\delta^3(\vec{k} - \vec{k}') \delta_{\lambda,\lambda'} V^{-+}_{k,k',\lambda,\lambda'}(\vec{p},\vec{p}')\right]$$

$$= \frac{2}{(2\pi)^3} \int d^3p' \sum_k V^{-+}_{k,k,\lambda,\lambda}(\vec{p}_\sigma,\vec{p}')$$

$$\textit{Substitute} \quad V^{-+}_{k,k,\lambda,\lambda'}(\vec{p},\vec{p}') \quad (3b), \sum_k \to \int \frac{d^3k}{(2\pi)^3}$$

$$= \frac{2}{(2\pi)^6} \int d^3p' d^3k \frac{(2\pi)^3 e^2 \hbar^2}{V4m} \frac{\vec{\varepsilon}_\lambda(\vec{k}) \cdot \vec{\varepsilon}_\lambda(\vec{k}')}{\omega(\vec{k})} \delta^3\left(\vec{k} - \vec{k} + \frac{\vec{p}}{\hbar} - \frac{\vec{p}_\sigma}{\hbar}\right)$$

$$= \frac{2}{(2\pi)^6} \int d^3p' d^3k \frac{(2\pi)^3 e^2 \hbar^2}{V4m} \frac{1}{\hbar ck} \delta^3\left(\frac{\vec{p}}{\hbar} - \frac{\vec{p}_\sigma}{\hbar}\right)$$

$$= \frac{e^2 \hbar}{2mcV} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k} \to \infty$$

I see then that the first-order energy shift is infinite! This is because the electron carries with it an infinite number of photons in the vacuum, and cannot be interpreted as an offset from some reference potential.