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Quantum Mechanics 3 Homework 5

1) Consider a linear chain of atoms of mass $m = 1$ joined by springs of constant k with a "site" potential $H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2}\kappa(q_{i+1} - q_i)^2 + \frac{R^2}{2} q_i^2$.

a) Obtain and solve the equations of motion and the dispersion relation of lattice waves: $\omega(k)$.

Hamilton's equations of motion are given by $\frac{\partial H}{\partial p_l} = \dot{q}_l \quad \frac{\partial H}{\partial q_l} = -\dot{p}_l$.

Then, I have:

$$\frac{\partial H}{\partial p_l} = \dot{q}_l \quad \frac{\partial H}{\partial q_l} = -\dot{p}_l$$

These give:

$$\frac{\partial H}{\partial p_l} = p_l = \dot{q}_l$$
$$\frac{\partial H}{\partial q_l} = -\dot{p}_l = -\ddot{q}_l = R^2 q_l + \kappa(2q_l - q_{l-1} - q_{l+1})$$

Making the assumption that $q_l = \alpha_k e^{ikla} e^{-i\omega_k t}$, then, I have:

$$\omega_k^2 q_l = R^2 q_l + \kappa\left(2q_l - q_l e^{-ika} - q_l e^{ika}\right)$$
$$\omega_k^2 = R^2 + \kappa\left(2 - e^{-ika} - e^{ika}\right) = R^2 + 4\kappa \sin^2\left(\frac{ak}{2}\right)$$

b) Why does $\omega(k) \to const \neq 0$ for $k \to 0$? Explain the difference between this and acoustic phonons.

From part a, I see that $\omega_k^2 \to R^2$ since $\sin^2\left(\frac{ak}{2}\right) \to 0$ as $k \to 0$, and so $\omega_k \to R$.

Physically, the reason for this is simple: the translational invariance has been broken by the introduction of the q_l^2 term, and now it costs energy to translate the lattice from equilibrium.

2) **<u>The Klein-Gordon scalar field theory</u>: Consider the Hamiltonian of problem 1, and consider the linear chain of N atoms, separated in equilibrium by a lattice spacing a and assume that the spring constant $\kappa = \frac{c^2}{a^2}$ with c constant.**

a) **Show that in the limit $a \to 0$ the dispersion relation is the same as for relativistic particles of mass $M^2 = \frac{R^2}{c^4}$ if c is the speed of light.**

Substituting for the spring constant,

$$\omega_k^2 = R^2 + 4\kappa \sin^2\left(\frac{ak}{2}\right) = R^2 + 4\frac{c^2}{a^2}\sin^2\left(\frac{ak}{2}\right)$$

In the limit $a \to 0$, then, taking the Taylor expansion I get:

$$4\frac{c^2}{a^2}\sin^2\left(\frac{ak}{2}\right) \approx c^2k^2 - \frac{1}{12}c^2k^4a^2 + \ldots$$

so

$$\omega_k^2 \approx R^2 + c^2k^2$$

In the relativistic case,

$$\begin{aligned} E &= \sqrt{p^2c^2 + M^2c^4} \to \sqrt{p^2c^2 + R^2} \\ \hbar\omega &= \sqrt{\hbar^2k^2c^2 + R^2} \\ \hbar^2\omega^2 &= \hbar^2k^2c^2 + R^2 \end{aligned}.$$

This is the same relation since in problem 1 I set $\hbar = 1$.

b) **For an atom with coordinate label i there corresponds a coordinate value $x = ia$, (then $i+1 \to x+a$). Define a field $\phi(x) = \frac{q_i}{\sqrt{a}}$ and a canonical momentum $\Pi(x) = \frac{p_i}{\sqrt{a}}$. Argue that in the limit $a \to 0$ with $Na = L =$ length of chain and $\sum_{i=1}^{N} a \to \int_0^L dx$ then $[\Pi(x), \phi(x')] = -i\hbar\delta(x - x')$ and that in this limit ($a \to 0$) that $H = \int_0^L dx\left[\frac{\Pi^2(x)}{2} + c^2\left(\frac{d\phi(x)}{dx}\right)^2 + M^2c^4\phi^2(x)\right]$. This is the Klein-Gordon field theory!**

First, take:

$$\int_0^L \Pi(x) \to \sum_{i=1}^{N} p_i = -i\hbar\vec{\nabla}_x \qquad \phi(x') \to \sum_{j=1}^{N} q_j$$

$$\int_0^L\int_0^L dxdx'\left[\Pi(x),\phi(x')\right] \to \sum_{i=1}^{N}\sum_{j=1}^{N}\left[p_i, q_j\right] = -i\hbar\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{i,j} \to -i\hbar\int_0^L\int_0^L dxdx'\,\delta(x-x')$$

$$\therefore \left[\Pi(x),\phi(x')\right] = -i\hbar\delta(x-x')$$

Next, take:

$$H = \sum_{i=1}^{N}\frac{p_i^2}{2} + \frac{1}{2}\kappa(q_{i+1}-q_i)^2 + \frac{R^2}{2}q_i^2$$

$$H = \sum_{i=1}^{N} a\left(\frac{p_i^2}{2a} + \frac{1}{2a}\kappa(q_{i+1}-q_i)^2 + \frac{R^2}{2a}q_i^2\right)$$

$Note:$

$$\sum_{i=1}^{N} a\left(\frac{p_i^2}{2a}\right) = \int_0^L \frac{\Pi^2(x)}{2}$$

$$\sum_{i=1}^{N} a\left(\frac{R^2 q_i^2}{2a}\right) = \int_0^L \frac{R^2\phi^2(x)}{2} = \int_0^L \frac{M^2c^4\phi^2(x)}{2}$$

$$\frac{1}{2}\kappa\sum_{i=1}^{N} a\,\frac{1}{a}(q_{i+1}-q_i)^2 = \frac{1}{2}\kappa\int_0^L \frac{1}{a}\left(\sqrt{a}\phi(x+a)-\sqrt{a}\phi(x)\right)^2 = \frac{1}{2}c^2\int_0^L \frac{1}{a^2}\left(\phi(x+a)-\phi(x)\right)^2$$

$$= \frac{1}{2}c^2\int_0^L\left(\frac{\phi(x+a)-\phi(x)}{a}\right)^2 =_{a\to 0} \frac{1}{2}c^2\int_0^L\left(\frac{d\phi(x)}{dx}\right)^2$$

$$\therefore H \to \int_0^L dx\left[\frac{\Pi^2(x)}{2} + c^2\left(\frac{d\phi(x)}{dx}\right)^2 + M^2c^4\phi^2(x)\right]$$

I have used the definition of derivative above, in the limit $a \to 0$.

3) <u>Quantization of the Electromagnetic Field</u>:

a) Show explicitly that $\vec{E}$ and $\vec{B}$ are invariant under the gauge transformations.

Recall that in any gauge, $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$.

Now define some gauge function Λ. Under the gauge transformation, then:

$$\vec{A} \to \vec{A} - \vec{\nabla}\Lambda$$

$$\phi \to \phi + \frac{\partial \Lambda}{\partial t}$$

So that:

$$\vec{B} \to \vec{\nabla} \times \left(\vec{A} - \vec{\nabla}\Lambda\right) = \vec{\nabla} \times \vec{A} - \vec{\nabla} \times \vec{\nabla}\Lambda = \vec{\nabla} \times \vec{A}$$ since $\vec{\nabla} \times \vec{\nabla} f = 0$, as expected.

Further,

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\left(\phi + \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A} - \vec{\nabla}\Lambda\right)$$

$$= -\vec{\nabla}\phi - \vec{\nabla}\frac{\partial \Lambda}{\partial t} - \frac{\partial \vec{A}}{\partial t} + \frac{\partial}{\partial t}\vec{\nabla}\Lambda = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

Just as expected.

b) From the inhomogeneous Maxwell equations with $\rho, \vec{J} \neq 0$, obtain the equations of motion for the scalar and vector potentials $\phi, \vec{A}$.

The inhomogeneous Maxwell's Equations are:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Rewriting these in terms of ϕ and $\vec{A}$, however, I see that the Gauss's equation for magnetic fields as well as Faraday's law are naturally satisfied and the other equations become:

$$\vec{B} = \vec{\nabla} \times \vec{A} \text{ and } \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}.$$

Thus:

$$-\vec{\nabla}^2\phi - \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right) = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla}^2\vec{A} = \mu_0 \vec{J} - \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\vec{\nabla}\phi - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

c) Consider that the vector potential $\vec{A}_o(\vec{x}, t)$ is such that $\vec{\nabla} \cdot \vec{A}_o(\vec{x}, t) \neq 0$. Show that one can always fund a function $\Lambda(\vec{x}, t)$ so that after the gauge transformation $\vec{A}_n(\vec{x}, t) = \vec{A}_o(\vec{x}, t) - \vec{\nabla}\Lambda(\vec{x}, t)$. The "new" vector potential $\vec{A}_n$

obeys the Coulomb gauge condition $\vec{\nabla}\cdot\vec{A}_n(\vec{x},t)=0$. Find $\Lambda(\vec{x},t)$ explicitly if you know $\vec{A}_o(\vec{x},t)$.

After such a transformation,

$$\vec{A}_n(\vec{x},t)=\vec{A}_o(\vec{x},t)-\vec{\nabla}\Lambda(\vec{x},t)$$
$$\vec{\nabla}\cdot\vec{A}_n(\vec{x},t)=\vec{\nabla}\cdot\vec{A}_o(\vec{x},t)-\vec{\nabla}^2\Lambda(\vec{x},t)$$
$$\therefore\vec{\nabla}^2\Lambda(\vec{x},t)=-\vec{\nabla}\cdot\vec{A}_o(\vec{x},t)$$

This is analogous to Poisson's equation; using the Green's function to find the Gauge function, then:

$$\Lambda(\vec{x},t)=\int\frac{d^3x'}{|\vec{x}-\vec{x}'|}\left[\vec{\nabla}\cdot\vec{A}_o(\vec{x}',t)\right]$$

d) **Obtain the equations obeyed by $\phi,\vec{A}$ in this gauge.**

Taking my result from part (b) and replacing $\vec{\nabla}\cdot\vec{A}=0$, I have:

$$-\vec{\nabla}^2\phi=\frac{\rho}{\varepsilon_0}$$
$$-\vec{\nabla}^2\vec{A}=\mu_0\vec{J}-\varepsilon_0\mu_0\frac{\partial}{\partial t}\vec{\nabla}\phi-\varepsilon_0\mu_0\frac{\partial^2\vec{A}}{\partial t^2}$$

e) **For $\rho,\vec{J}=0$ (no sources), find the solutions of the equations of motion under the Coulomb gauge condition for $\vec{A}$ and ϕ and the dispersion relations of free electromagnetic waves.**

Then recalling that $\varepsilon_0\mu_0=\frac{1}{c^2}$:

$$\vec{\nabla}^2\phi=0$$
$$\vec{\nabla}^2\vec{A}=\frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\phi+\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$$

Typically, one would have chosen the Lorenz gauge where the equations would decouple easily. From the first equation, it is clear that ϕ satisfies Laplace's equation. There are a great deal of forms for these solutions; The most typical form in free space, then, is the Spherical version indexed {n, l, m}. No specific constraint on the time-dependence is indicated yet, and I give them in terms of Legendre polynomials $P_n(r)$ and Spherical harmonics $Y_m^l(\theta,\phi)$.

$$\phi_{nlm} = f_{nlm}(t)P_n(r)Y_m^l(\theta,\phi)$$

Now consider the boundary condition where $\phi(r \to \infty) = 0$. Since the Legendre polynomials do not converge at infinity, this eliminates the n degree of freedom and so solutions must be like:

$$\phi_{lm} = f_{lm}(t)Y_m^l(\theta,\phi)$$

Finally, if $\phi(r \to \infty) = 0$ at all times t, then $f_{nlm}(t) = a_{nlm} + b_{nlm}t = 0$ and $\phi_{lm} = 0$. Now I have:

$$\phi = 0$$

$$\vec{\nabla}^2 \vec{A} = \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$

$\vec{A}$ then has the form of a plane wave: $\vec{A} = C_1 \vec{\varepsilon} e^{i\vec{k}\cdot\vec{x}} e^{-i\omega_k t} + C_2 \vec{\varepsilon} e^{-i\vec{k}\cdot\vec{x}} e^{i\omega_k t}$.

Note the constraint on $\vec{\varepsilon}$ obtained as such:

$$\vec{\nabla}\cdot\vec{A} = 0 \to \vec{\varepsilon}\cdot\vec{\nabla}\psi = 0 \to \vec{k}\cdot\vec{\varepsilon} = 0 \to \vec{k} \perp \vec{\varepsilon}$$

Then:

$$-\vec{k}^2\vec{A} = -\frac{\omega_k^2}{c^2}\vec{A}$$

$$\omega_k^2 = \vec{k}^2 c^2$$

Again, c^2 is free here and no specific rate of propagation is set for $\vec{A}$. The speed of light can be fixed for electric and magnetic fields, however, by taking Maxwell's equations in source-free space and obtaining a dispersion relation in terms of the coefficients ε_0, μ_0.

4) **The Hamiltonian for free Electromagnetic waves is** $H = \int d^3x \left[\frac{1}{2}\vec{E}^2(\vec{x},t) + \frac{1}{2}\vec{B}^2(\vec{x},t)\right]$**.**

a) **From the solution of $\vec{A}$ and ϕ in the Coulomb gauge found in problem #3, write $\vec{A}$ as a combination** $\vec{A}(\vec{x},t) = \frac{1}{\sqrt{V}}\sum_k\sum_\lambda \vec{\varepsilon}_\lambda(k)\left[\alpha_{k,\lambda}e^{i(\vec{k}\cdot\vec{x}-\omega(k)t)} + \alpha^*{}_{k,\lambda}e^{-i(\vec{k}\cdot\vec{x}-\omega(k)t)}\right]$ **and put this into H. Find the correct normalization of $\alpha_{k,\lambda}$ so that the Hamiltonian**

$$H = \sum_k\sum_\lambda \hbar\omega(k)\left[\hat{a}^+_{k,\lambda}\hat{a}_{k,\lambda} + \frac{1}{2}\right].$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$Let \quad \omega(k) \to \omega_\lambda(k)$$

$$\vec{E}(\vec{x},t) = -\frac{1}{c\sqrt{V}}\sum_k\sum_\lambda i\omega(k)\vec{\varepsilon}_\lambda(k)\left[-\alpha_{k,\lambda}e^{i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)} + \alpha^*{}_{k,\lambda}e^{-i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}\right]$$
$$\therefore \vec{E}^2(\vec{x},t)$$
$$= -\frac{1}{Vc^2}\sum_{k,k'}\sum_{\lambda,\lambda'}\omega(k)\omega(k')\varepsilon_\lambda(k)\varepsilon_{\lambda'}(k')\left[-\alpha_{k,\lambda}e^{i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)} + \alpha^*{}_{k,\lambda}e^{-i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}\right]\left[-\alpha_{k',\lambda'}e^{i\left(\vec{k}'\cdot\vec{x}-\omega_\lambda(k')t\right)} + \alpha^*{}_{k',\lambda'}e^{-i\left(\vec{k}'\cdot\vec{x}-\omega_\lambda(k')t\right)}\right]$$

$$\vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\sum\vec{\varepsilon}_\lambda(k)\psi = \vec{\nabla}\psi\times\vec{\varepsilon}_\lambda(k) + \psi\vec{\nabla}\times\vec{\varepsilon}_\lambda(k) = \vec{\nabla}\psi\times\vec{\varepsilon}_\lambda(k)$$
$$= \frac{1}{\sqrt{V}}\sum_k\sum_\lambda i\vec{k}\left[\alpha_{k,\lambda}e^{i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)} - \alpha^*{}_{k,\lambda}e^{-i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}\right]\times\vec{\varepsilon}_\lambda(k)$$

In the Coulomb gauge, $\vec{k} \perp \vec{\varepsilon}$ so that:

$$\therefore \vec{B}^2(\vec{x},t)$$
$$= -\frac{1}{V}\sum_{k,k'}\sum_{\lambda,\lambda'}kk'\varepsilon_\lambda(k)\varepsilon_{\lambda'}(k')\left[\alpha_{k,\lambda}e^{i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)} - \alpha^*{}_{k,\lambda}e^{-i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}\right]\left[\alpha_{k',\lambda'}e^{i\left(\vec{k}'\cdot\vec{x}-\omega_\lambda(k')t\right)} - \alpha^*{}_{k',\lambda'}e^{-i\left(\vec{k}'\cdot\vec{x}-\omega_\lambda(k')t\right)}\right]$$

Now consider the full Hamiltonian: $H = \int d^3x\left[\frac{1}{2}\vec{E}^2(\vec{x},t) + \frac{1}{2}\vec{B}^2(\vec{x},t)\right]$. Here I have many terms like:

$$\therefore \vec{B}^2(\vec{x},t)$$
$$\int d^3x\begin{bmatrix} c_{k,k';\lambda,\lambda';\alpha\alpha}e^{i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}e^{i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)} + c_{k,k';\lambda,\lambda';\alpha\alpha^+}e^{i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}e^{-i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)} \\ + c_{k,k';\lambda,\lambda';\alpha^+\alpha}e^{-i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}e^{i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)} + c_{k,k';\lambda,\lambda';\alpha^+\alpha^+}e^{-i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}e^{-i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}\end{bmatrix}$$

These functions are orthogonal unless the exponent of one is exactly -1 times the other. Then:

$$\int d^3x\begin{bmatrix} c_{k,k';\lambda,\lambda';\alpha\alpha}e^{i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}e^{i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)} + c_{k,k';\lambda,\lambda';\alpha\alpha^+}e^{i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}e^{-i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)} \\ + c_{k,k';\lambda,\lambda';\alpha^+\alpha}e^{-i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}e^{i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)} + c_{k,k';\lambda,\lambda';\alpha^+\alpha^+}e^{-i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}e^{-i\left(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t\right)}\end{bmatrix}$$
$$= \left[0 + c_{k,k';\lambda,\lambda';\alpha\alpha^+}\delta_{k,k'}\delta_{\lambda,\lambda'} + c_{k,k';\lambda,\lambda';\alpha^+\alpha}\delta_{k,k'}\delta_{\lambda,\lambda'} + 0\right]$$

The delta-function on lambda arises from two orthogonal polarizations of the plane wave.

Now:

$$H = \int d^3x\left[\frac{1}{2}\vec{E}^2(\vec{x},t) + \frac{1}{2}\vec{B}^2(\vec{x},t)\right]$$

$$= \sum_{k,\lambda}\left[\frac{1}{2}\frac{\omega_\lambda{}^2(k)}{c^2}\varepsilon_\lambda{}^2(k)\left(\alpha_{k,\lambda}\alpha_{k,\lambda}{}^* + \alpha_{k,\lambda}{}^*\alpha_{k,\lambda}\right) + \frac{1}{2}k^2\varepsilon_\lambda{}^2(k)\left(\alpha_{k,\lambda}\alpha_{k,\lambda}{}^* + \alpha_{k,\lambda}{}^*\alpha_{k,\lambda}\right)\right]$$

$$Because \quad \omega_\lambda{}^2(k) = k^2c^2, \varepsilon_\lambda{}^2(k) = 1$$

$$= \sum_{k,\lambda}\left[\frac{\omega_\lambda{}^2(k)}{c^2}\left(\alpha_{k,\lambda}\alpha_{k,\lambda}{}^* + \alpha_{k,\lambda}{}^*\alpha_{k,\lambda}\right)\right]$$

$$\sqrt{\frac{2\omega_\lambda(k)}{\hbar c^2}}\alpha_{k,\lambda} \to \hat{a}_{k,\lambda}$$

$$= \sum_{k,\lambda}\left[\frac{\hbar\omega_\lambda(k)}{2}\left(\hat{a}_{k,\lambda}\hat{a}^+{}_{k,\lambda} + \hat{a}_{k,\lambda}{}^+\hat{a}_{k,\lambda}\right)\right]$$

b) **Quantize the E&M field by imposing $\left[\hat{a}_{k,\lambda}, \hat{a}^+{}_{k,\lambda}\right] = \delta_{k,k'}\delta_{\lambda,\lambda'}$, and find the commutators $\left[E_i(\vec{x},t), A_j(\vec{x}',t')\right]$. When do these vanish? Why? Explain the physics of your answer.**

$$H = \sum_{k,\lambda}\left[\frac{\hbar\omega_\lambda(k)}{2}\left(\hat{a}_{k,\lambda}\hat{a}^+{}_{k,\lambda} + \hat{a}_{k,\lambda}{}^+\hat{a}_{k,\lambda}\right)\right]$$

Using the imposed commutator, then, I have:

$$H = \sum_{k,\lambda}\left[\frac{\hbar\omega_\lambda(k)}{2}\left(2\hat{a}_{k,\lambda}{}^+\hat{a}_{k,\lambda} + 1\right)\right] = \sum_{k,\lambda}\left[\hbar\omega_\lambda(k)\left(\hat{a}_{k,\lambda}{}^+\hat{a}_{k,\lambda} + \frac{1}{2}\right)\right]$$

From part a,

$$\vec{E}_{k,\lambda}(\vec{x},t) = -\frac{1}{c\sqrt{V}} i\omega_\lambda(k)\vec{\varepsilon}_\lambda(k)\left[-\alpha_{k,\lambda}e^{i(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t)} + \alpha^*{}_{k,\lambda}e^{-i(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t)}\right]$$

$$\vec{A}_{l,\lambda'}(\vec{x},t) = \frac{1}{\sqrt{V}}\vec{\varepsilon}_{\lambda'}(l)\left[\alpha_{l,\lambda'}e^{i(\vec{l}\cdot\vec{x}-\omega_{\lambda'}(l)t)} + \alpha^*{}_{l,\lambda'}e^{-i(\vec{l}\cdot\vec{x}-\omega_{\lambda'}(l)t)}\right]$$

$$\left[\vec{E}_{k,\lambda}(\vec{x},t),\vec{A}_{l,\lambda'}(\vec{x}',t')\right] = -\frac{i\omega(k)\vec{\varepsilon}_\lambda(k)\cdot\vec{\varepsilon}_{\lambda'}(l)}{cV}\left(-e^{i(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t)}e^{-i(\vec{l}\cdot\vec{x}'-\omega_{\lambda'}(l)t')}\left[\alpha_{k,\lambda},\alpha^*{}_{l,\lambda'}\right] + e^{-i(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t)}e^{i(\vec{l}\cdot\vec{x}'-\omega_{\lambda'}(l)t')}\left[\alpha^*{}_{k,\lambda},\alpha_{l,\lambda'}\right]\right)$$

$$= -\frac{i\omega_\lambda(k)\vec{\varepsilon}_\lambda(k)\cdot\vec{\varepsilon}_{\lambda'}(l)}{cV}\begin{pmatrix} -e^{i(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t')}e^{-i(\vec{l}\cdot\vec{x}'-\omega_{\lambda'}(l)t')}\dfrac{\hbar c^2}{2\omega_\lambda(k)}\delta_{k,l}\delta_{\lambda,\lambda'} \\ -e^{-i(\vec{k}\cdot\vec{x}-\omega_\lambda(k)t)}e^{i(\vec{l}\cdot\vec{x}'-\omega_{\lambda'}(l)t')}\dfrac{\hbar c^2}{2\omega_\lambda(k)}\delta_{k,l}\delta_{\lambda,\lambda'} \end{pmatrix}$$

$$= \frac{i\hbar c}{2V}\left(e^{i(\vec{k}\cdot(\vec{x}-\vec{x}')-\omega_\lambda(k)(t-t'))}\delta_{k,l}\delta_{\lambda,\lambda'} + e^{-i(\vec{k}\cdot(\vec{x}-\vec{x}')-\omega_\lambda(k)(t-t'))}\delta_{k,l}\delta_{\lambda,\lambda'}\right)$$

or

$$\left[\vec{E}_k(\vec{x},t),\vec{A}_l(\vec{x}',t')\right] = \left[\sum_\lambda \vec{E}_{k,\lambda}(\vec{x},t),\sum_l \vec{A}_{l,\lambda}(\vec{x}',t')\right] = \frac{i\hbar c}{V}\left(e^{i(\vec{k}\cdot(\vec{x}-\vec{x}')-\omega_\lambda(k)(t-t'))} + e^{-i(\vec{k}\cdot(\vec{x}-\vec{x}')-\omega_\lambda(k)(t-t'))}\delta_{k,l}\right)\delta_{k,l}$$

$$= \frac{2i\hbar c}{V}\cos\left(\vec{k}\cdot(\vec{x}-\vec{x}')-\omega_\lambda(k)(t-t')\right)\delta_{k,l}$$

So I see that this commutator vanishes whenever the electric field is unrelated index-wise to this term of $\vec{A}$; this is hardly surprising. The electric and magnetic fields are well-known to be subject to superposition, and so I see that when an electric field encounters an area of vector potential like the one from which it originated that it “sees” it and otherwise it is simply superposed. Further, I see that it is heavily dependent on whether this electric field is or out of phase with the associated vector potential, vanishing when they are 90 degrees out of phase.