

1) Coherent States in Many-Body Systems:

Consider $\psi(\vec{x})$ a second quantized Bosonic operator and a vacuum state $|0\rangle$; $\psi(\vec{x})|0\rangle = 0$. Consider a complex function $\phi(\vec{x})$ with a Fourier transform defined as $\alpha_{\vec{p}} = \frac{1}{\sqrt{N}} \int d^3x e^{i\frac{\vec{p}\cdot\vec{x}}{\hbar}} \phi(\vec{x})$.

Expand $\psi(\vec{x})$ into single free particle states with wavefunctions $\frac{1}{\sqrt{N}} e^{i\frac{\vec{p}\cdot\vec{x}}{\hbar}}$.

A many body coherent state is given by

$$|\alpha_{\vec{p}_1}, \dots, \alpha_{\vec{p}_N}\rangle = e^{\int d^3x [\psi^\dagger(\vec{x})\phi^*(\vec{x}) - \psi(\vec{x})\phi(\vec{x})]} |0\rangle$$

a) Compute the overlap of this state with the Fock state of free particles

$$|n_{\vec{p}_1}, \dots, n_{\vec{p}_N}\rangle.$$

By the definition of coherent state, $a_{\vec{p}_i} |\alpha_{\vec{p}_i}\rangle = \alpha_{\vec{p}_i} |\alpha_{\vec{p}_i}\rangle$. That being said, finding the amplitude $\langle n_{\vec{p}_1}, \dots, n_{\vec{p}_N} | \alpha_{\vec{p}_1}, \dots, \alpha_{\vec{p}_N} \rangle$, I have:

$$\begin{aligned} \langle n_{\vec{p}_1}, \dots, n_{\vec{p}_N} | &= \langle 0 | \left[\sum_{j=0}^N \frac{1}{\sqrt{n_{\vec{p}_j}!}} a_{\vec{p}_j}^{n_{\vec{p}_j}} \right] \\ \langle n_{\vec{p}_1}, \dots, n_{\vec{p}_N} | \alpha_{\vec{p}_1}, \dots, \alpha_{\vec{p}_N} \rangle &= \langle 0 | \left[\sum_{j=0}^N \frac{1}{\sqrt{n_{\vec{p}_j}!}} a_{\vec{p}_j}^{n_{\vec{p}_j}} \right] | \alpha_{\vec{p}_1}, \dots, \alpha_{\vec{p}_N} \rangle \\ &= \sum_{j=0}^N \frac{1}{\sqrt{n_{\vec{p}_j}!}} \langle 0 | a_{\vec{p}_j}^{n_{\vec{p}_j}} | \alpha_{\vec{p}_1}, \dots, \alpha_{\vec{p}_N} \rangle = \sum_{j=0}^N \frac{\alpha_{\vec{p}_j}^{n_{\vec{p}_j}}}{\sqrt{n_{\vec{p}_j}!}} \langle 0 | \alpha_{\vec{p}_1}, \dots, \alpha_{\vec{p}_N} \rangle \end{aligned}$$

Now all that's left is to find the value of the projection $\langle 0 | \alpha_{\vec{p}_1}, \dots, \alpha_{\vec{p}_N} \rangle$: without a relation on $[\psi(\vec{x}), \psi^\dagger(\vec{x})]$, this is a rather difficult thing to find. Supposing that $[\psi(\vec{x}), \psi^\dagger(\vec{x}')] = \delta(\vec{x} - \vec{x}')$, I write:

$$|\alpha_{\vec{p}_1}, \dots, \alpha_{\vec{p}_N}\rangle = e^{\int d^3x [\psi^\dagger(\vec{x})\phi^*(\vec{x}) - \psi(\vec{x})\phi(\vec{x})]} |0\rangle = e^{\left[\int d^3x \psi^\dagger(\vec{x})\phi^*(\vec{x}) - \int d^3x \psi(\vec{x})\phi(\vec{x}) \right]} |0\rangle:$$

Using the identity $[\hat{A}, \hat{B}] = C$, $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0 \rightarrow e^{(\hat{A}+\hat{B})} = e^{\hat{A}} e^{\hat{B}} e^{\frac{-1}{2}C}$, I can write:

$$e^{\left[\int d^3x \psi^+(\bar{x}) \phi^*(\bar{x}) - \int d^3x' \psi^+(\bar{x}') \phi^*(\bar{x}') \right]} |0\rangle$$

$$* = e^{\int d^3x \psi^+(\bar{x}) \phi^*(\bar{x})} e^{-\int d^3x' \psi^+(\bar{x}') \phi^*(\bar{x}')} e^{-\frac{1}{2} \left[\int d^3x \psi^+(\bar{x}) \phi^*(\bar{x}) - \int d^3x' \psi^+(\bar{x}') \phi^*(\bar{x}') \right]} |0\rangle$$

$$\begin{aligned} \text{Lemma : } & \left[\int d^3x \psi^+(\bar{x}) \phi^*(\bar{x}), -\int d^3x' \psi^+(\bar{x}') \phi^*(\bar{x}') \right] \\ & = -\left[\int d^3x \psi^+(\bar{x}) \phi^*(\bar{x}), \int d^3x' \psi^+(\bar{x}') \phi^*(\bar{x}') \right] \\ & = -\int d^3x d^3x' \psi^+(\bar{x}) \phi^*(\bar{x}) \psi^+(\bar{x}') \phi^*(\bar{x}') + \int d^3x d^3x' \psi^+(\bar{x}') \phi^*(\bar{x}') \psi^+(\bar{x}) \phi^*(\bar{x}) \\ & = -\int d^3x d^3x' \psi^+(\bar{x}) \phi^*(\bar{x}) \psi^+(\bar{x}') \phi^*(\bar{x}') + \int d^3x d^3x' (\psi^+(\bar{x}) \psi^+(\bar{x}') + \delta(\bar{x} - \bar{x}')) \phi^*(\bar{x}') \phi^*(\bar{x}) \\ & = \int d^3x d^3x' \delta(\bar{x} - \bar{x}') \phi^*(\bar{x}') \phi^*(\bar{x}) = \int d^3x |\phi^*(\bar{x})|^2 \end{aligned}$$

$$\text{Continue : } * = e^{i \int d^3x \psi^+(\bar{x}) \phi^*(\bar{x})} e^{-i \int d^3x' \psi^+(\bar{x}') \phi^*(\bar{x}')} e^{-\frac{1}{2} \int d^3x |\phi^*(\bar{x})|^2} |0\rangle$$

$$** = e^{-\frac{1}{2} C \int d^3x |\phi^*(\bar{x})|^2} e^{\int d^3x \psi^+(\bar{x}) \phi^*(\bar{x})} e^{-\int d^3x' \psi^+(\bar{x}') \phi^*(\bar{x}')} |0\rangle$$

$$\text{Lemma : } \psi^+(\bar{x}') |0\rangle = 0, \therefore e^{-\int d^3x' \psi^+(\bar{x}') \phi^*(\bar{x}')} = 1 + 0 + 0 + 0 \dots$$

$$\text{Continue : } ** = e^{-\frac{1}{2} \int d^3x |\phi^*(\bar{x})|^2} e^{\int d^3x \psi^+(\bar{x}) \phi^*(\bar{x})} |0\rangle$$

Lemma : $\psi^+(\bar{x})$: creation – only, because $\psi^+(\bar{x}) |0\rangle = 0$ annihilation only

$$\text{Continue : } ** = \langle 0 | e^{-\frac{1}{2} C \int d^3x |\phi^*(\bar{x})|^2} e^{\int d^3x \psi^+(\bar{x}) \phi^*(\bar{x})} |0\rangle = \langle 0 | e^{-\frac{1}{2} \int d^3x |\phi^*(\bar{x})|^2} |0\rangle = e^{-\frac{1}{2} \int d^3x |\phi^*(\bar{x})|^2}$$

Therefore, $\langle 0 | \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} \rangle = e^{-\frac{1}{2} \int d^3x |\phi^*(\bar{x})|^2}$ because $[\psi^+(\bar{x}), \psi^+(\bar{x}')] = \delta(\bar{x} - \bar{x}')$ and

$$\begin{aligned} \langle n_{\bar{p}_1}, \dots, n_{\bar{p}_N} | \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} \rangle & = \sum_{j=0}^N \frac{\alpha_{\bar{p}_j}^{n_{\bar{p}_j}}}{\sqrt{n_{\bar{p}_j}!}} \langle 0 | \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} \rangle \\ & = e^{-\frac{1}{2} \int d^3x |\phi^*(\bar{x})|^2} \sum_{j=0}^N \frac{\alpha_{\bar{p}_j}^{n_{\bar{p}_j}}}{\sqrt{n_{\bar{p}_j}!}} \quad \text{if } [\psi^+(\bar{x}), \psi^+(\bar{x}')] = \delta(\bar{x} - \bar{x}') \end{aligned}$$

b) Compute the average number of particles of momentum \bar{p} in the coherent state.

One way to do this would be to actually let the number operator $\hat{N}_{\bar{p}} = a_{\bar{p}}^+ a_{\bar{p}}$ act.

However, I see that I've already got the job largely complete! Taking my result from part (a), I should have:

$$\begin{aligned} \langle \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} | \hat{N}_{\bar{p}} | \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} \rangle &= \sum_{n=0}^{\infty} n \langle \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} | n_{\bar{p}} \rangle^2 \\ &= e^{-\int d^3x |\phi(\bar{x})|^2} \sum_{n=0}^{\infty} \langle \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} | \frac{\alpha_{\bar{p}}^{*n}}{\sqrt{n!}} n \frac{\alpha_{\bar{p}}^n}{\sqrt{n!}} | \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} \rangle \end{aligned}$$

Note that I have written the projector on the first line above with only one element translated into Fock space—this is avoid N infinite sums over Fock space particle counts.

What I've done is taken the projection from part (a), squared the amplitude to get the probability, then added the eigenvalue of n. What I'm left with is:

$$\begin{aligned} \langle \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} | \hat{N}_{\bar{p}} | \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} \rangle &= e^{-\int d^3x |\phi(\bar{x})|^2} \sum_{n=0}^{\infty} n \langle \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} | \frac{\alpha_{\bar{p}}^{*n}}{\sqrt{n!}} \frac{\alpha_{\bar{p}}^n}{\sqrt{n!}} | \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} \rangle \\ &= e^{-\int d^3x |\phi(\bar{x})|^2} \sum_{n=0}^{\infty} n \frac{|\alpha_{\bar{p}}^n|^2}{n!} = e^{-\int d^3x |\phi(\bar{x})|^2} \sum_{n=1}^{\infty} \frac{|\alpha_{\bar{p}}^n|^2}{(n-1)!} \\ \text{where } [\psi(\bar{x}), \psi^+(\bar{x}')] &= \delta(\bar{x} - \bar{x}') \end{aligned}$$

c) What is $\langle \alpha | \psi(\bar{x}) | \alpha \rangle$ and $\langle \alpha | \psi^+(\bar{x}) | \alpha \rangle$?

Assuming the identity $[\psi(\bar{x}), \psi^+(\bar{x}')] = \delta(\bar{x} - \bar{x}')$, I have (using the identity $[\hat{A}, e^{\alpha \hat{B}}] = \alpha [\hat{A}, \hat{B}] e^{\alpha \hat{B}}$):

$$\begin{aligned} \psi(\bar{x}) | \alpha \rangle &= \psi(\bar{x}) e^{\int d^3x' [\psi^+(\bar{x}') \phi^*(\bar{x}') - \psi(\bar{x}') \phi(\bar{x}')] | 0 \rangle} \\ &= [\psi(\bar{x}), \int d^3x' [\psi^+(\bar{x}') \phi^*(\bar{x}') - \psi(\bar{x}') \phi(\bar{x}')] e^{\int d^3x' [\psi^+(\bar{x}') \phi^*(\bar{x}') - \psi(\bar{x}') \phi(\bar{x}')] | 0 \rangle} + e^{\int d^3x' [\psi^+(\bar{x}') \phi^*(\bar{x}') - \psi(\bar{x}') \phi(\bar{x}')] | 0 \rangle} \psi(\bar{x}) | 0 \rangle \\ &= [\psi(\bar{x}), \int d^3x' \psi^+(\bar{x}') \phi^*(\bar{x}')] | \alpha \rangle = \int d^3x' \delta^3(\bar{x} - \bar{x}') \phi^*(\bar{x}') | \alpha \rangle = \phi^*(\bar{x}) | \alpha \rangle \\ \langle \alpha | \psi(\bar{x}) | \alpha \rangle &= \phi^*(\bar{x}) \end{aligned}$$

By a similar procedure, $\langle \alpha | \psi^+(\bar{x}) | \alpha \rangle = \phi(\bar{x})$, where a minus-sign has come from the exponential and the reversed commutator.

2) Coherent States versus Bose-Einstein Condensates:

Consider the case of problem 1 above and take $\alpha_{\bar{p}=0} \equiv \alpha$ and $\alpha_{\bar{p} \neq 0} = 0$, this corresponds to $\phi(\bar{x}) = \text{const} = \phi_0$.

a) Find α_0 so that $\langle \alpha_0 | \hat{N}_{\bar{p}=0} | \alpha_0 \rangle = N$

Taking $[\psi(\bar{x}), \psi^+(\bar{x}')] = \delta(\bar{x} - \bar{x}')$ and dropping the normalization term so that I can renormalize more conveniently with the denominator, I have (from part a):

$$\frac{\langle \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} | \hat{N}_{\bar{p}} | \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} \rangle}{\langle \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} | \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} \rangle} = \frac{\sum_{n=1}^{\infty} \frac{|\alpha_{\bar{p}=0}^n|^2}{(n-1)!}}{\sum_{n=1}^{\infty} \frac{|\alpha_{\bar{p}=0}^n|^2}{(n)!}} = \frac{|a_0|^2 e^{|a_0|^2}}{e^{|a_0|^2}} = |\alpha_0|^2 = N_0$$

$$\sqrt{N_0} = |\alpha_0|$$

or

$$\alpha_0 = \sqrt{N_0} e^{i\varphi}$$

$$\varphi \text{ arbitrary } [0, 2\pi]$$

b) Give the coefficients of the expansion $|\alpha_0\rangle = \sum_{n=0}^{\infty} c_n |n_{\bar{p}=0}\rangle$.

From problem 1, I recall that:

$$\begin{aligned} & \langle n_{\bar{p}_1}, \dots, n_{\bar{p}_N} | \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} \rangle \\ &= e^{-\int d^3x |\phi(\bar{x})|^2} \sum_{j=0}^N \frac{\alpha_{\bar{p}_j}^{n_{\bar{p}_j}}}{\sqrt{n_{\bar{p}_j}!}} \quad \text{if } [\psi(\bar{x}), \psi^+(\bar{x}')] = \delta(\bar{x} - \bar{x}') \end{aligned}$$

and so in this case,

$$\begin{aligned} \langle n_{\bar{p}=0} | \alpha_0 \rangle &= e^{-\int d^3x |\phi(\bar{x})|^2} \frac{\alpha_{\bar{p}=0}^{n_{\bar{p}=0}}}{\sqrt{n_{\bar{p}=0}!}} \\ |\alpha_0\rangle &= e^{-\int d^3x |\phi(\bar{x})|^2} \sum_{n=0}^{\infty} \frac{\alpha_{\bar{p}=0}^{n_{\bar{p}=0}}}{\sqrt{n_{\bar{p}=0}!}} |n_{\bar{p}=0}\rangle \end{aligned}$$

As in part a, it is more convenient to drop this normalization term and renormalize later, but this is equally clear.

c) Compute $\langle \alpha_0 | \hat{N}_{\bar{p}=0}^2 | \alpha_0 \rangle$ and the variance $\left[\langle \alpha_0 | \hat{N}_{\bar{p}=0}^2 | \alpha_0 \rangle - \langle \alpha_0 | \hat{N}_{\bar{p}=0} | \alpha_0 \rangle^2 \right]^{1/2} = \Delta$

and show that $\frac{\Delta}{\langle \alpha_0 | \hat{N}_{\bar{p}=0} | \alpha_0 \rangle} \rightarrow 0$ as $\langle \alpha_0 | \hat{N}_{\bar{p}=0} | \alpha_0 \rangle = N_0 \rightarrow \infty$.

Note: in the limit $N_0 \rightarrow \infty$ their state (coherent) describes very well a Bose-Einstein condensate $|n_{\bar{p}=0}, 0, 0, \dots\rangle$.

Using the same procedure as in problem 1b, I have:

$$\frac{\langle \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} | \hat{N}_{\bar{p}=0}^2 | \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} \rangle}{\langle \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} | \alpha_{\bar{p}_1}, \dots, \alpha_{\bar{p}_N} \rangle} = \frac{\sum_{n=1}^{\infty} n \frac{|\alpha_{\bar{p}=0}^n|^2}{(n-1)!}}{e^{|\alpha_0|^2}} = |\alpha_0|^2 (1 + |\alpha_0|^2)$$

where $[\psi(\bar{x}), \psi^+(\bar{x}')] = \delta(\bar{x} - \bar{x}')$

$$\left[|\alpha_0|^2 (1 + |\alpha_0|^2) - |\alpha_0|^4 \right]^{\frac{1}{2}} = \Delta$$

$$= \left[|\alpha_0|^2 \right]^{\frac{1}{2}} = |\alpha_0| = \sqrt{N_0}$$

Clearly, $\frac{\sqrt{N_0}}{N_0} = \frac{1}{\sqrt{N_0}} \rightarrow 0$ as $N_0 \rightarrow \infty$.

3) The second quantized momentum operator is $\bar{P} = \int d^3x \psi^+(\bar{x})(-i\hbar \bar{\nabla}_x) \psi(\bar{x})$.

a) Obtain $[\bar{P}, \psi(\bar{x})]$ and use the result to show that $e^{-\frac{i}{\hbar} \bar{P} \cdot \bar{a}} \psi(\bar{x}) e^{\frac{i}{\hbar} \bar{P} \cdot \bar{a}} = \psi(\bar{x} + \bar{a})$ with \bar{a} a constant vector.

Recall from assignment #1 that:

$$e^{\hat{A} \hat{B}} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \dots + \frac{1}{n!} [\hat{A}, \dots (n \text{ commutators}), \dots, [\hat{A}, \hat{B}]]$$

Assume: $[\psi(\bar{x}), \psi^+(\bar{x}')] = \delta^3(\bar{x} - \bar{x}')$

$$\begin{aligned} [\bar{P}, \psi(\bar{x}')] &= \int d^3x \psi^+(\bar{x})(-i\hbar \bar{\nabla}_x) \psi(\bar{x}) \psi(\bar{x}') - \psi(\bar{x}') \int d^3x \psi^+(\bar{x})(-i\hbar \bar{\nabla}_x) \psi(\bar{x}) \\ &= -i\hbar \int d^3x \left[-\delta^3(\bar{x}' - \bar{x}) + \psi(\bar{x}') \psi^+(\bar{x}) \bar{\nabla}_x \psi(\bar{x}) \right] - \psi(\bar{x}') \int d^3x \psi^+(\bar{x})(-i\hbar \bar{\nabla}_x) \psi(\bar{x}) \end{aligned}$$

Cancelling:

$$= i\hbar \int d^3x \delta^3(\bar{x}' - \bar{x}) \bar{\nabla}_x \psi(\bar{x}) = i\hbar \bar{\nabla}_x \psi(\bar{x}) \Big|_{\bar{x}=\bar{x}'}$$

In that case:

$$\begin{aligned}
& e^{-\frac{i}{\hbar}\bar{P}\cdot\bar{a}} \psi(\bar{x}') e^{\frac{i}{\hbar}\bar{P}\cdot\bar{a}} \\
& \left[-\frac{i}{\hbar} \bar{P} \cdot \bar{a}, \psi(\bar{x}') \right] = -\frac{i}{\hbar} [\bar{P}, \psi(\bar{x}')] \cdot \bar{a} = \bar{\nabla}_x \psi(\bar{x}) \Big|_{\bar{x}=\bar{x}'} \cdot \bar{a} \\
& = \psi(\bar{x}') + \left[-\frac{i}{\hbar} \bar{P} \cdot \bar{a}, \psi(\bar{x}') \right] + \frac{1}{2!} \left[-\frac{i}{\hbar} \bar{P} \cdot \bar{a}, \left[-\frac{i}{\hbar} \bar{P} \cdot \bar{a}, \psi(\bar{x}') \right] \right] + \dots \text{etc...} \\
& = \psi(\bar{x}') + \bar{\nabla}_x \psi(\bar{x}) \Big|_{\bar{x}=\bar{x}'} \cdot \bar{a} + \frac{1}{2!} \bar{\nabla}_x^2 \psi(\bar{x}) \Big|_{\bar{x}=\bar{x}'} (\bar{a})^2 + \frac{1}{3!} \bar{\nabla}_x^3 \psi(\bar{x}) \Big|_{\bar{x}=\bar{x}'} \cdot \bar{a} (\bar{a})^2 + \frac{1}{4!} \bar{\nabla}_x^4 \psi(\bar{x}) \Big|_{\bar{x}=\bar{x}'} (\bar{a})^4 \dots \text{etc...} \\
& = \psi(\bar{x}' + \bar{a})
\end{aligned}$$

Note that in this formulation the dot product is intermittently necessary due to the alternating vector and scalar result of $\bar{\nabla}_x^n$, but nonetheless the form of the result is still identical to that of a Taylor expansion.

b) Consider the Hamiltonian

$$\hat{H} = \int d^3x \psi^+(\bar{x}) \left[-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) + \int d^3x d^3y \psi^+(\bar{x}) \psi^+(\bar{y}) V(|\bar{x} - \bar{y}|) \psi(\bar{y}) \psi(\bar{x})$$

and compute $e^{-\frac{i}{\hbar}\bar{P}\cdot\bar{a}} \hat{H} e^{\frac{i}{\hbar}\bar{P}\cdot\bar{a}}$.

$$[\bar{P}, \hat{H}] = \left[\bar{P}, \int d^3x \psi^+(\bar{x}) \left[-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) \right] + \left[\bar{P}, \int d^3x d^3y \psi^+(\bar{x}) \psi^+(\bar{y}) V(|\bar{x} - \bar{y}|) \psi(\bar{y}) \psi(\bar{x}) \right]$$

Solving this one portion at a time, I have:

$$\begin{aligned}
& \left[\bar{P}, \int d^3x \psi^+(\bar{x}) \left[-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) \right] \\
& = \left[\bar{P}, \int d^3x \psi^+(\bar{x}) \left[-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) \right] \\
& = * \bar{P} \int d^3x \psi^+(\bar{x}) \left[-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) - ** \int d^3x \psi^+(\bar{x}) \left[-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) \bar{P}
\end{aligned}$$

For now, just consider a single portion of the expression:

$$\begin{aligned}
& * \bar{P} \int d^3x \psi^+(\bar{x}) \left[-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) \\
& = \int d^3x \left([\bar{P}, \psi^+(\bar{x})] + \psi^+(\bar{x}) \bar{P} \right) \left[-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) \\
& = \int d^3x \left(\begin{aligned} & \left[\bar{P}, \psi^+(\bar{x}) \right] \left[-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) \\ & + \psi^+(\bar{x}) \left(\left[\bar{P}, -\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) + \left(-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right) \left([\bar{P}, \psi(\bar{x})] + \psi(\bar{x}) \bar{P} \right) \right) \end{aligned} \right)
\end{aligned}$$

Now subtracting the other integral marked ** off, I have:

$$\begin{aligned}
& \left[\bar{P}, \int d^3x \psi^+(\bar{x}) \left[-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) \right] \\
& = \int d^3x \left(\begin{aligned} & \left[\bar{P}, \psi^+(\bar{x}) \right] \left[-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) + \psi^+(\bar{x}) \left[\bar{P}, -\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) \\ & + \psi^+(\bar{x}) \left(-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right) [\bar{P}, \psi(\bar{x})] \end{aligned} \right)
\end{aligned}$$

Since the momentum operator necessarily commutes with itself, I can eliminate the kinetic portion of the commutator in the second term:

$$*** = \int d^3x \left(\left[\bar{P}, \psi^+(\bar{x}) \right] \left[-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right] \psi(\bar{x}) + \psi^+(\bar{x}) [\bar{P}, U(\bar{x})] \psi(\bar{x}) + \psi^+(\bar{x}) \left(-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right) [\bar{P}, \psi(\bar{x})] \right)$$

Lemma:

$$\begin{aligned}
& \int d^3x \psi^+(\bar{x}) \left(-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right) [\bar{P}, \psi(\bar{x})] = \int d^3x \left([\psi^+(\bar{x}), \bar{P}] \left(-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right) \psi(\bar{x}) \right)^+ \\
& = \left(-\int d^3x [\bar{P}, \psi^+(\bar{x})] \left(-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right) \psi(\bar{x}) \right)^+
\end{aligned}$$

However, since the Hamiltonian is a real-valued function:

$$\begin{aligned}
& = -\int d^3x [\bar{P}, \psi^+(\bar{x})] \left(-\frac{\hbar^2 \bar{\nabla}_x^2}{2m} + U(\bar{x}) \right) \psi(\bar{x}) \\
& \therefore *** = \int d^3x \psi^+(\bar{x}) [\bar{P}, U(\bar{x})] \psi(\bar{x}).
\end{aligned}$$

Now solving the other half:

$$\begin{aligned}
& \left[\bar{P}, \int d^3x d^3y \psi^+(\bar{x}) \psi^+(\bar{y}) V(|\bar{x} - \bar{y}|) \psi(\bar{y}) \psi(\bar{x}) \right] \\
&= \int d^3x d^3y \left[\bar{P}, \psi^+(\bar{x}) \right] \psi^+(\bar{y}) V(|\bar{x} - \bar{y}|) \psi(\bar{y}) \psi(\bar{x}) + \int d^3x d^3y \psi^+(\bar{x}) \left[\bar{P}, \psi^+(\bar{y}) \right] V(|\bar{x} - \bar{y}|) \psi(\bar{y}) \psi(\bar{x}) \\
&+ \int d^3x d^3y \psi^+(\bar{x}) \psi^+(\bar{y}) V(|\bar{x} - \bar{y}|) \left[\bar{P}, \psi(\bar{y}) \right] \psi(\bar{x}) + \int d^3x d^3y \psi^+(\bar{x}) \psi^+(\bar{y}) V(|\bar{x} - \bar{y}|) \psi(\bar{y}) \left[\bar{P}, \psi(\bar{x}) \right] \\
&+ \int d^3x d^3y \psi^+(\bar{x}) \psi^+(\bar{y}) \left[\bar{P}, V(|\bar{x} - \bar{y}|) \right] \psi(\bar{y}) \psi(\bar{x}) \\
&= \int d^3x d^3y \psi^+(\bar{x}) \psi^+(\bar{y}) \left[\bar{P}, V(|\bar{x} - \bar{y}|) \right] \psi(\bar{y}) \psi(\bar{x}) \quad (\text{use lemma above})
\end{aligned}$$

where I have used the commutator identity:

$$\left[\hat{A}, \prod_i \hat{B}_i \right] = \sum_i \left(\prod_{j < i} \hat{B}_j \right) \left[\hat{A}, \hat{B}_i \right] \left(\prod_{j > i} \hat{B}_j \right)$$

Note that in order to invoke said lemma, it was necessary to assume that the function V was a symmetric, Hermitian operator (which it should be, given the form of the argument). This indicates that:

$$\left[\bar{P}, \hat{H} \right] = \int d^3x \psi^+(\bar{x}) \left[\bar{P}, U(\bar{x}) \right] \psi(\bar{x}) + \int d^3x d^3y \psi^+(\bar{x}) \psi^+(\bar{y}) \left[\bar{P}, V(|\bar{x} - \bar{y}|) \right] \psi(\bar{y}) \psi(\bar{x})$$

use lemma above:

$$\int d^3x d^3y \psi^+(\bar{x}) \psi^+(\bar{y}) \left[\frac{i}{\hbar} \bar{P}, V(|\bar{x} - \bar{y}|) \right] \psi(\bar{y}) \psi(\bar{x}) \rightarrow 0$$

because $V(|\bar{x} - \bar{y}|)$ Symmetric

so

$$\left[\bar{P}, \hat{H} \right] = \int d^3x \psi^+(\bar{x}) \left[\bar{P}, U(\bar{x}) \right] \psi(\bar{x})$$

$$e^{\frac{i}{\hbar} \bar{P} \cdot \bar{a}} \hat{H} e^{\frac{i}{\hbar} \bar{P} \cdot \bar{a}} =$$

$$\hat{H} + \left(\left[-\frac{i}{\hbar} \bar{P}, U(\bar{x}) \right] \right) \cdot \bar{a} + \frac{1}{2!} \left(\left[-\frac{i}{\hbar} \bar{P}, \left[-\frac{i}{\hbar} \bar{P}, U(\bar{x}) \right] \right] \right) \bar{a}^2 + \frac{1}{3!} \left(\left[-\frac{i}{\hbar} \bar{P}, \left[-\frac{i}{\hbar} \bar{P}, \left[-\frac{i}{\hbar} \bar{P}, U(\bar{x}) \right] \right] \right] \right) \bar{a}^3 + \dots$$

c) When is the total momentum conserved? $[\bar{P}, \hat{H}] = 0$? Explain why.

In order to take the final steps above, I had to use the assumption that the potential V was a symmetric function of x and y (which is clearly the case from the form of the function). Physically speaking, this is the condition that forces between particles are symmetric.

Further, I see that there is a contribution from $[\bar{P}, U(\bar{x})]$: indeed, this “external potential” should be constant in order to conserve momentum so that $[\bar{P}, \hat{H}] = 0$ (seen in the final portion of part b).

What have we learned here? One, that for conservation of momentum to hold, the potential energy in the system should not come from some source external to the system (the condition $[\bar{P}, U(\bar{x})] = 0$), and that forces should be symmetric, for the condition that

$V(|\bar{x} - \bar{y}|)$ is, as shown, a symmetric function of \bar{x} and \bar{y} . Further, by an analog of the lemma above, this I saw that the commutator $[\bar{P}, V(|\bar{x} - \bar{y}|)] = 0$ went to zero when integrated over all space by symmetry.

4) Fermi pressure in the relativistic free electron gas: Consider a free gas of relativistic electrons with $\varepsilon(\bar{k}) = \sqrt{(\hbar\bar{k}c)^2 + m^2c^4}$ and Hamiltonian

$$\hat{H} = \sum_{i,\sigma} \varepsilon(\bar{k}_i) a_{i,\sigma}^+ a_{i,\sigma}, \text{ and there are } N \text{ electrons.}$$

a) Find the Fermi momentum k_F as a function of density.

The free electron gas wave-function is well known to be eigenfunctions of the del-squared operator:

$$\psi(\bar{x}) = \sum_{n_x, n_y, n_z = -\infty}^{\infty} C_{n_x, n_y, n_z} e^{i\frac{n_x\pi}{L}x} e^{i\frac{n_y\pi}{L}y} e^{i\frac{n_z\pi}{L}z}$$

I see, then, that the allowed energies and momenta are indexed n_x, n_y, n_z and so appear in three-space as points inside a sphere. Further, with electrons I take $g_0 = 2$.

Then integrating concentric spheres in momentum space up to the Fermi level, I have:

$$\frac{N}{V} = \frac{g_0}{(2\pi)^3} \int_0^{k_F} 4\pi k^2 dk = \frac{g_0}{3 \cdot 2\pi^2} k_F^3$$

$$k_F = \left(\frac{6\pi^2}{g_0} \frac{N}{V} \right)^{\frac{1}{3}} = \left(3\pi^2 \frac{N}{V} \right)^{\frac{1}{3}}$$

b) Find the total energy of the ground state: rescale the integral and carry it out in terms of the dimensionless variable $X = \frac{\hbar k}{mc}$.

Integrating over the sphere of the Fermi sea of momenta the relativistic energy associated with these momenta, I will be left with the Fermi energy:

$$\begin{aligned}
\frac{\mathcal{E}_{tot}}{V} &= \frac{g_0}{(2\pi)^3} \int_0^{k_F} \mathcal{E}(k) 4\pi k^2 dk = \frac{g_0}{(2\pi)^3} \int_0^{k_F} \sqrt{(\hbar kc)^2 + m^2 c^4} 4\pi k^2 dk \\
&= \frac{g_0 m c^2}{2\pi^2} \int_0^{k_F} k^2 \sqrt{\left(\frac{\hbar k}{mc}\right)^2 + 1} dk = \frac{g_0 m^4 c^5}{2\pi^2 \hbar^3} \int_0^{k_F} \left(\frac{\hbar k}{mc}\right)^2 \sqrt{\left(\frac{\hbar k}{mc}\right)^2 + 1} \frac{\hbar}{mc} dk \\
x_F &= \frac{\hbar k}{mc} \quad dx_F = \frac{\hbar}{mc} dk \\
&= \frac{g_0 m^4 c^5}{2\pi^2 \hbar^3} \int_0^{\frac{\hbar k_F}{mc}} x_F^2 \sqrt{x_F^2 + 1} dx_F \\
\frac{\mathcal{E}_{tot}}{V} &= \frac{g_0 m^4 c^5}{2\pi^2 \hbar^3} \left[\frac{1}{8} \left(\sqrt{1 + \left(\frac{\hbar k_F}{mc}\right)^2} \left(\frac{\hbar k_F}{mc} + 2 \left(\frac{\hbar k_F}{mc}\right)^3 \right) - \text{ArcSinh}\left(\frac{\hbar k_F}{mc}\right) \right) \right]
\end{aligned}$$

with $g_0 = 2, k_F = \left(3\pi^2 \frac{N}{V}\right)^{\frac{1}{3}}$.

c) Find the Fermi degeneracy pressure. Show that it is of the form

$$p = \frac{mc^2}{\lambda_e^3} \Phi(x_F), \text{ with } \lambda_e = \frac{\hbar}{mc} \text{ the Compton wavelength of the electron and}$$

$$\Phi \text{ a dimensionless function of its argument and } x_F = \frac{\hbar k_F}{mc}.$$

Note that:

$$\frac{\partial x_F}{\partial V} = \frac{\partial}{\partial V} \left(\frac{\hbar k_F}{mc} \right) = \frac{\hbar}{mc} \frac{1}{3} \left(3\pi^2 \frac{N}{V} \right)^{-\frac{2}{3}} \left(-3\pi^2 \frac{N}{V^2} \right) = -\frac{\hbar}{3mcV} \left(3\pi^2 \frac{N}{V} \right)^{\frac{1}{3}} = -\frac{1}{3V} \frac{\hbar k_F}{mc} = -\frac{1}{3V} x_F$$

Further,

$$p = -\frac{\partial \mathcal{E}_{tot}}{\partial V} = -\frac{\partial}{\partial V} \left(V \frac{\mathcal{E}_{tot}}{V} \right) = -\frac{\mathcal{E}_{tot}}{V} - V \frac{\partial}{\partial V} \left(\frac{\mathcal{E}_{tot}}{V} \right)$$

Evaluating this derivative in Mathematica, I get:

$$\begin{aligned}
p &= \frac{m^4 c^5}{\hbar^3} \left[\frac{g_0}{16\pi^2} \left(\left(\frac{2}{3} x_F^3 - x_F \right) \sqrt{1 + x_F^2} + \text{ArcSinh}(x_F) \right) \right] \\
&= \frac{mc^2}{\lambda_e^3} \Phi(x_F)
\end{aligned}$$

$$\Phi(x_F) = \frac{g_0}{16\pi^2} \left(\left(\frac{2}{3} x_F^3 - x_F \right) \sqrt{1 + x_F^2} + \text{ArcSinh}(x_F) \right)$$

d) Give the limits $x_F \gg 1$ (relativistic) and $x_F \ll 1$, show that for a fixed electron density the pressure is lower in the relativistic case. This is the origin of the maximum mass for White Dwarfs, the Chandrasekhar limit.

Taking

$$p = \frac{mc^2}{\lambda_e^3} \left[\frac{g_0}{16\pi^2} \left(\left(\frac{2}{3} x_F^3 - x_F \right) \sqrt{1 + x_F^2} + \text{ArcSinh}(x_F) \right) \right],$$

I see that at very small x, then (Taylor expanding with Mathematica):

$$p_{x_F \approx 0} = \frac{mc^2}{\lambda_e^3} \frac{g_0}{16\pi^2} \left[\frac{8}{15} x_F^5 - \frac{4}{21} x_F^7 + \dots \right]$$

At very large x:

$$p_{\frac{1}{x_F} \approx 0} = \frac{mc^2}{\lambda_e^3} \frac{g_0}{16\pi^2} \left[\frac{2}{3} x_F^4 - \frac{2}{3} x_F^2 + \dots \right]$$

And so I see that the relativistic case grows more slowly than the classical case. This is the origin of the Chandrasekhar limit.