Ben Sauerwine
Quantum Mechanics 3 Homework 2

1) **Consider the general 2-particle Fock state $|2\rangle = \int d^3x_1 d^3x_2\chi(\vec{x}_1,\vec{x}_2)\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)|0\rangle$ where $|0\rangle$ is the vacuum state and $\chi(x_1,x_2)$ is a complex function. Consider a first-quantized Hamiltonian $H_x = -\frac{\hbar^2\vec{\nabla}_x^2}{2m}+U(\vec{x})$ and $\hat{H}_x = \int d^3x\psi^+(\vec{x})H_x\psi(\vec{x})$, $\hat{N} = \int d^3x\psi^+(\vec{x})\psi(\vec{x})$. Show that:**

a) $\hat{N}|2\rangle = 2|2\rangle$

$$\int d^3x\psi^+(\vec{x})\psi(\vec{x})|2\rangle = \int d^3x\psi^+(\vec{x})\psi(\vec{x})\int d^3x_1d^3x_2\chi(\vec{x}_1,\vec{x}_2)\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)|0\rangle$$

$$\pm\begin{matrix}Bose\\ Fermi\end{matrix}$$

$$\psi(\vec{x})\psi^+(\vec{y}) = \delta(\vec{x}-\vec{y})\pm\psi^+(\vec{y})\psi(\vec{x}) \quad \psi(\vec{x})|0\rangle = 0 \quad \psi^+(\vec{x})\psi^+(\vec{y}) = \pm\psi^+(\vec{y})\psi^+(\vec{x})$$

$$= \int d^3x\psi^+(\vec{x})\int d^3x_1d^3x_2\chi(\vec{x}_1,\vec{x}_2)\left(\delta(\vec{x}-\vec{x}_1)\pm\psi^+(\vec{x}_1)\psi(\vec{x})\right)\psi^+(\vec{x}_2)|0\rangle$$

$$= \int d^3x\psi^+(\vec{x})\int d^3x_1d^3x_2\chi(\vec{x}_1,\vec{x}_2)\left(\delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)\pm\psi^+(\vec{x}_1)\left(\delta(\vec{x}-\vec{x}_2)\pm\psi^+(\vec{x}_2)\psi(\vec{x})\right)\right)|0\rangle$$

$$= \int d^3x\psi^+(\vec{x})\int d^3x_1d^3x_2\chi(\vec{x}_1,\vec{x}_2)\left(\delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)\pm\psi^+(\vec{x}_1)\left(\delta(\vec{x}-\vec{x}_2)\pm\psi^+(\vec{x}_2)\psi(\vec{x})\right)\right)|0\rangle$$

$$= \int d^3x_1d^3x_2\int d^3x\psi^+(\vec{x})\chi(\vec{x}_1,\vec{x}_2)\left(\delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)\pm\psi^+(\vec{x}_1)\delta(\vec{x}-\vec{x}_2)\right)|0\rangle$$

$$= \int d^3x_1d^3x_2\int d^3x\psi^+(\vec{x})\chi(\vec{x}_1,\vec{x}_2)\delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)|0\rangle$$

$$\pm\int d^3x_1d^3x_2\int d^3x\delta(\vec{x}-\vec{x}_2)\psi^+(\vec{x})\chi(\vec{x}_1,\vec{x}_2)\psi^+(\vec{x}_1)|0\rangle$$

$$= \int d^3x_1d^3x_2\chi(\vec{x}_1,\vec{x}_2)\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)|0\rangle$$

$$\pm\int d^3x_1d^3x_2\int d^3\vec{x}\delta(\vec{x}-\vec{x}_2)\psi^+(\vec{x})\chi(\vec{x}_1,\vec{x}_2)\psi^+(\vec{x}_1)|0\rangle$$

$$= |2\rangle \pm\int d^3x_1d^3x_2\psi^+(\vec{x}_2)\chi(\vec{x}_1,\vec{x}_2)\psi^+(\vec{x}_1)|0\rangle = |2\rangle\pm\left(\pm|2\rangle\right) = 2|2\rangle$$

b) If $\hat{H}_x|2\rangle = E|2\rangle$ **then** $H\chi(x_1,x_2) = E\chi(x_1,x_2)$**.**

$$\pm\begin{matrix}Bose\\ Fermi\end{matrix}$$

$$\psi(\vec{x})\psi^+(\vec{y}) = \delta(\vec{x}-\vec{y})\pm\psi^+(\vec{y})\psi(\vec{x}) \quad \psi(\vec{x})|0\rangle = 0 \quad \psi^+(\vec{x})\psi^+(\vec{y}) = \pm\psi^+(\vec{y})\psi^+(\vec{x})$$

$$\hat{H}|2\rangle = \int d^3x\psi^+(\vec{x})H_x\psi(\vec{x})\int d^3x_1d^3x_2\chi(\vec{x}_1,\vec{x}_2)\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)|0\rangle$$

$$\{EXPRESSION \quad A\} = \int d^3x\psi^+(\vec{x})H_x\int d^3x_1d^3x_2\chi(\vec{x}_1,\vec{x}_2)\begin{bmatrix}\delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)\\ \pm\delta(\vec{x}-\vec{x}_2)\psi^+(\vec{x}_1)\end{bmatrix}|0\rangle$$

I have simplified with the procedure from part (a). Now I would like to commute the Hamiltonian through these delta functions so that they may collapse against the outermost integrals. The complex part may certainly pass, but the divergence is a bit more complex. Consider just the first Hamiltonian acting on the just the top entry in the array from Expression A above:

$$=\int d^3x\,\psi^+(\vec{x})\int d^3x_1 d^3x_2 H_x \delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$

$$=*\int d^3x\,\psi^+(\vec{x})\int d^3x_1 d^3x_2 U(\vec{x})\delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$

$$-\frac{\hbar^2}{2m}**\int d^3x\,\psi^+(\vec{x})\int d^3x_1 d^3x_2 \left(\vec{\nabla}_{x_1}{}^2\delta(\vec{x}-\vec{x}_1)\right)\psi^+(\vec{x}_2)\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$

First, I evaluate the simpler, * portion:

$$*=\int d^3x\,\psi^+(\vec{x})\int d^3x_1 d^3x_2 U_1(x)\delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$

$$=\int d^3x_1 d^3x_2 U_1(x)\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$

$$=\int d^3x_1 d^3x_2 U_1(x)\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)U(\vec{x}_1)\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$

where I have used the result of part (a) above. Next, I integrate by parts the ** portion:

$$**=\int d^3x\,\psi^+(\vec{x})\int d^3x_1 d^3x_2 \left(\vec{\nabla}_x{}^2\delta(\vec{x}-\vec{x}_1)\right)\psi^+(\vec{x}_2)\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$

$$consider \quad \int d^3x_1 d^3x_2 \left(\vec{\nabla}_x{}^2\delta(\vec{x}-\vec{x}_1)\right)\psi^+(\vec{x}_2)\chi(\vec{x}_1,\vec{x}_2)$$

$$\vec{\nabla}_x{}^2\delta(\vec{x}-\vec{x}_1)=(-1)(-1)\vec{\nabla}_{x_1}{}^2\delta(\vec{x}-\vec{x}_1)$$

$$=\int d^3x_1 d^3x_2 \left(\vec{\nabla}_{x_1}{}^2\delta(\vec{x}-\vec{x}_1)\right)\psi^+(\vec{x}_2)\chi(\vec{x}_1,\vec{x}_2)$$

$$u=\psi^+(\vec{x}_2)\chi(\vec{x}_1,\vec{x}_2) \qquad dv=\vec{\nabla}_{x_1}{}^2\delta(\vec{x}-\vec{x}_1)d^3x_1$$

$$du=\psi^+(\vec{x}_2)\vec{\nabla}_{x_1}\chi(\vec{x}_1,\vec{x}_2)d^3x_1 \qquad v=-\vec{\nabla}_{x_1}\delta(\vec{x}-\vec{x}_1)$$

$$=-\psi^+(\vec{x}_2)\chi(\vec{x}_1,\vec{x}_2)\vec{\nabla}_{x_1}\delta(\vec{x}-\vec{x}_1)\Big|_{x_1\to\infty \quad \lim its}+\int d^3x_1\vec{\nabla}_{x_1}\delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)\vec{\nabla}_{x_1}\chi(\vec{x}_1,\vec{x}_2)$$

$$Normality\rightarrow\int d^3x_1\vec{\nabla}_{x_1}\delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)\vec{\nabla}_{x_1}\chi(\vec{x}_1,\vec{x}_2)$$

$$u=\psi^+(\vec{x}_2)\vec{\nabla}_{x_1}\chi(\vec{x}_1,\vec{x}_2) \qquad dv=\vec{\nabla}_{x_1}\delta(\vec{x}-\vec{x}_1)d^3x_1$$

$$du=\psi^+(\vec{x}_2)\vec{\nabla}_{x_1}{}^2\chi(\vec{x}_1,\vec{x}_2)d^3x_1 \qquad v=-\delta(\vec{x}-\vec{x}_1)$$

$$=-\psi^+(\vec{x}_2)\vec{\nabla}_{x_1}\chi(\vec{x}_1,\vec{x}_2)\delta(\vec{x}-\vec{x}_1)\Big|_{x_1\to\infty \quad \lim its}+\int d^3x_1\delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)\vec{\nabla}_{x_1}{}^2\chi(\vec{x}_1,\vec{x}_2)$$

$$Normality\rightarrow\int d^3x_1\delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)\vec{\nabla}_{x_1}{}^2\chi(\vec{x}_1,\vec{x}_2)$$

$$**=\int d^3x\,\psi^+(\vec{x})\int d^3x_1 d^3x_2\delta(\vec{x}-\vec{x}_1)\psi^+(\vec{x}_2)\vec{\nabla}_{x_1}{}^2\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$

$$=\int d^3x_1 d^3x_{21}\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)\vec{\nabla}_{x_1}{}^2\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$

An analogous procedure on Expression A above using the lower column will give:

$$\pm\begin{array}{c}Bose\\Fermi\end{array}$$

$$\pm\int d^3x\,\psi^+(\vec{x})H_x\int d^3x_1d^3x_2\,\chi(\vec{x}_1,\vec{x}_2)\delta(\vec{x}-\vec{x}_2)\psi^+(\vec{x}_1)|0\rangle$$
$$=\pm\int d^3x_1d^3x_2\,\psi^+(\vec{x}_2)\psi^+(\vec{x}_1)\left(\vec{\nabla}^2_{x_2}+U(\vec{x}_2)\right)\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$
$$=\pm\pm\int d^3x_1d^3x_2\,\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)\left(\vec{\nabla}^2_{x_2}+U(\vec{x}_2)\right)\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$
$$=\int d^3x_1d^3x_2\,\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)\left(\vec{\nabla}^2_{x_2}+U(\vec{x}_2)\right)\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$

Combining the results from the upper and lower columns, I see that:

$$\hat{H}_x|2\rangle=E|2\rangle=\int d^3x_1d^3x_2\,\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)\left(-\frac{\hbar^2}{2m}\vec{\nabla}^2_{x_1}+U(\vec{x}_1)-\frac{\hbar^2}{2m}\vec{\nabla}^2_{x_2}+U(\vec{x}_2)\right)\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$
$$=E\int d^3x_1d^3x_2\,\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)\chi(\vec{x}_1,\vec{x}_2)|0\rangle$$

The only way that this can be true is if

$$\left(-\frac{\hbar^2}{2m}\vec{\nabla}^2_{x_1}+U(\vec{x}_1)-\frac{\hbar^2}{2m}\vec{\nabla}^2_{x_2}+U(\vec{x}_2)\right)\chi(\vec{x}_1,\vec{x}_2)=E\chi(\vec{x}_1,\vec{x}_2)$$

c) Show that $\chi(\vec{x}_1,\vec{x}_2)=\pm\chi(\vec{x}_2,\vec{x}_1)$ (+ for bosons, - for fermions).

$$\pm\begin{array}{c}Bose\\Fermi\end{array}$$

$$\psi^+(\vec{x})\psi^+(\vec{y})=\pm\psi^+(\vec{y})\psi^+(\vec{x})$$
$$\int d^3x_1d^3x_2\,\chi(\vec{x}_1,\vec{x}_2)\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)|0\rangle=\pm\int d^3x_1d^3x_2\,\chi(\vec{x}_1,\vec{x}_2)\psi^+(\vec{x}_2)\psi^+(\vec{x}_1)|0\rangle$$

Renaming the dummy indices, then, I may rename $\vec{x}_1\leftrightarrow\vec{x}_2$

$$=\pm\int d^3x_1d^3x_2\,\chi(\vec{x}_2,\vec{x}_1)\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)|0\rangle$$
$$\therefore\ \chi(\vec{x}_2,\vec{x}_1)=\pm\chi(\vec{x}_1,\vec{x}_2)$$

d) Generalize the results for N particles.

(i) $\hat{N}|N\rangle=N|N\rangle$

$$|N\rangle = \int \chi\left(\bigcup_{i=1}^{N}\vec{x}_i\right)\prod_{i=1}^{N} d^3x_i\psi^+(\vec{x}_i)|0\rangle$$

$$\hat{N}|N\rangle = \int d^3y\,\psi^+(\vec{y})\psi(\vec{y})\int \chi\left(\bigcup_{i=1}^{N}\vec{x}_i\right)\prod_{i=1}^{N} d^3x_i\psi^+(\vec{x}_i)|0\rangle$$

$$= \int d^3y\,\psi^+(\vec{y})\int \chi\left(\bigcup_{i=1}^{N}\vec{x}_i\right)\prod_{i=1}^{N} d^3x_k\sum_{j=1}^{N}\delta^3(\vec{y}-\vec{x}_j)(\pm)^{j-1}\prod_{k=1,k\neq j}^{N}\psi^+(\vec{x}_k)|0\rangle$$

$$= \int \chi\left(\bigcup_{i=1}^{N}\vec{x}_i\right)\prod_{i=1}^{N} d^3x_k\sum_{j=1}^{N}\psi^+(\vec{x}_j)(\pm)^{j-1}\prod_{k=1,k\neq j}^{N}\psi^+(\vec{x}_k)|0\rangle$$

$$= \sum_{j=1}^{N}(\pm)^{j-1}(\pm)^{j-1}\int \chi\left(\bigcup_{i=1}^{N}\vec{x}_i\right)\prod_{i=1}^{N} d^3x_k\prod_{k=1}^{N}\psi^+(\vec{x}_k)|0\rangle$$

$$= N|N\rangle$$

(ii) If $\hat{H}_y|N\rangle = E|N\rangle$, $\sum_i H_{x_i}\chi(\vec{x}_1,\vec{x}_2,...,\vec{x}_i) = E\chi(\vec{x}_1,\vec{x}_2,...,\vec{x}_i)$

Greatly simplifying the procedure from part (b), I start with:

$$H_y = \frac{-\hbar^2\vec{\nabla}_y^2}{2m} + U(\vec{y})$$

$$\hat{H}_y|N\rangle = \int d^3y\,\psi^+(\vec{y})H_y\psi(\vec{y})\int \chi\left(\bigcup_{i=1}^{N}\vec{x}_i\right)\prod_{i=1}^{N} d^3x_i\psi^+(\vec{x}_i)|0\rangle$$

$$= \int d^3y\,\psi^+(\vec{y})H_y\int \chi\left(\bigcup_{i=1}^{N}\vec{x}_i\right)\prod_{i=1}^{N} d^3x_k\sum_{j=1}^{N}\delta^3(\vec{y}-\vec{x}_j)(\pm)^{j-1}\prod_{k=1,k\neq j}^{N}\psi^+(\vec{x}_k)|0\rangle$$

I determined in part (b) that the kinetic parts of the Hamiltonian ultimately interact with the delta function, bringing only respective Hamiltonians for each particle. I also saw in part (b) that in order to maintain normal ordering, j – 1 commutations must occur so that I bring an additional ± factor, maintaining the desired overall sign.

$$\hat{H}|N\rangle = \int\left[H_i\chi\left(\bigcup_{i=1}^{N}\vec{x}_i\right)\right]\prod_{i=1}^{N} d^3x_i\psi^+(\vec{x}_i)|0\rangle$$

and so the $\left[\sum_i H_i\chi\left(\bigcup_{i=1}^{N}\vec{x}_i\right)\right] = E\chi\left(\bigcup_{i=1}^{N}\vec{x}_i\right)$ is the only way for the given $\hat{H}|N\rangle = E|N\rangle$ to be satisfied.

(iii) $\chi\left(\bigcup_{i=1}^{N}\vec{x}_i\right) = \pm\chi_{1\ \ swap}\left(\bigcup_{i=1}^{N}\vec{x}'_i\right)$ (+ for bosons, - for fermions).

Examine the results from part (c) to understand this explanation:

After a single swap, if I were to re-index as in part (c), I need to commute several of the creation operators lined up against the ket. Consider the two creation operators, in particular, that got swapped: there is some number n of operators between them, where n may be zero. I need to move each of these operators to the other's position. Then, each of the two operators will pass by each of the n operators between them once, for a $(\pm)^{2n} = 1$ factor (+ for bosons, - for fermions). Finally, they will have to pass through one another at some point: This brings one more $(\pm)$ factor. Then, overall, a single $(\pm)$ is brought from a single swap and $\chi\left(\bigcup_{i=1}^{N} \vec{x}_i\right) = \pm\chi_{1\ swap}\left(\bigcup_{i=1}^{N} \vec{x}'_i\right)$.

- **e) Consider $U \equiv 0$ (free particles) and show that the two particle wave function is obtained as $\langle 0|\psi(\vec{x}_1)\psi(\vec{x}_2)|1_{p_1}, 1_{p_2}\rangle$ where $|1_{p_1}, 1_{p_2}\rangle = a_{p_1}^{+} a_{p_2}^{+}|0\rangle$.**

$$\pm : \begin{matrix} \mathit{Bosons} \\ \mathit{Fermions} \end{matrix}$$

$$\begin{aligned}
&\langle 0|\psi(\vec{x}_1)\psi(\vec{x}_2) a_{p_1}^{+} a_{p_2}^{+}|0\rangle \\
&= \langle 0|\psi(\vec{x}_1)\psi(\vec{x}_2) a_{p_1}^{+} a_{p_2}^{+}|0\rangle \\
&= \langle 0|\psi(\vec{x}_1)\left[\delta^3\left(\vec{p}^{(2)} - \vec{p}_1\right) \pm a_{p_1}^{+}\psi(\vec{x}_2)\right] a_{p_2}^{+}|0\rangle \\
&= \langle 0|\left[\delta^3\left(\vec{p}^{(2)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(1)} - \vec{p}_2\right) \pm \delta^3\left(\vec{p}^{(1)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(2)} - \vec{p}_2\right)\right]|0\rangle
\end{aligned}$$

Fourier-transforming these back into real space, I have integrals like:

$$\int d^3 p e^{i\vec{x}\cdot\frac{\vec{p}}{\hbar}} \delta(\vec{p} - \vec{p}_1) = \int d^3 p e^{i\vec{x}\cdot\frac{\vec{p}}{\hbar}} \delta(\vec{p} - \vec{p}_1) = e^{\frac{i}{\hbar}\vec{x}\cdot\vec{p}}$$

which reproduces the desired wave functions, within a normalization factor.

- **2) Consider 3 free particles.**

- **a) Construct directly the Bosonic wave function for three particles with momenta $\vec{p}_1, \vec{p}_2, \vec{p}_3$ (namely, construct the permanent).**

For all three parts, define $\phi_i(\vec{x}) = e^{\frac{i}{\hbar}\vec{p}_i\cdot\vec{x}} a_i^{+}$.

$$|1_{\vec{p}_1}, 1_{\vec{p}_2}, 1_{\vec{p}_3}\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix} \phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\phi_3(\vec{x}_3) + \phi_1(\vec{x}_2)\phi_2(\vec{x}_1)\phi_3(\vec{x}_3) + \phi_1(\vec{x}_1)\phi_2(\vec{x}_3)\phi_3(\vec{x}_2) \\ + \phi_1(\vec{x}_3)\phi_2(\vec{x}_1)\phi_3(\vec{x}_2) + \phi_1(\vec{x}_3)\phi_2(\vec{x}_2)\phi_3(\vec{x}_1) + \phi_1(\vec{x}_2)\phi_2(\vec{x}_3)\phi_3(\vec{x}_1) \end{bmatrix}|0\rangle$$

- **b) Construct the Fermionic wave function from the Slater determinant.**

$$\frac{1}{\sqrt{6}}\begin{vmatrix}\phi_1(\vec{x}_1) & \phi_1(\vec{x}_2) & \phi_1(\vec{x}_3)\\ \phi_2(\vec{x}_1) & \phi_2(\vec{x}_2) & \phi_2(\vec{x}_3)\\ \phi_3(\vec{x}_1) & \phi_3(\vec{x}_2) & \phi_3(\vec{x}_3)\end{vmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}\phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\phi_3(\vec{x}_3) - \phi_1(\vec{x}_2)\phi_2(\vec{x}_1)\phi_3(\vec{x}_3)\\ -\phi_1(\vec{x}_1)\phi_2(\vec{x}_3)\phi_3(\vec{x}_2) + \phi_1(\vec{x}_3)\phi_2(\vec{x}_1)\phi_3(\vec{x}_2)\\ -\phi_1(\vec{x}_3)\phi_2(\vec{x}_2)\phi_3(\vec{x}_1) + \phi_1(\vec{x}_2)\phi_2(\vec{x}_3)\phi_3(\vec{x}_1)\end{bmatrix}\left|0\right\rangle$$

c) Obtain it from $\left\langle 0\left|\psi(\vec{x}_1)\psi(\vec{x}_2)\psi(\vec{x}_3)\right|1_{p_1},1_{p_2},1_{p_3}\right\rangle$ **where** $\left|1_{p_1},1_{p_2},1_{p_3}\right\rangle = a^+_{p_1}a^+_{p_2}a^+_{p_3}\left|0\right\rangle$**.**

Much as in 1e, I have:

$-:$ *Fermions*

$$\left\langle 0\left|\psi(\vec{x}_1)\psi(\vec{x}_2)\psi(\vec{x}_3)a^+_{p_1}a^+_{p_2}a^+_{p_3}\right|0\right\rangle$$

$$= \left\langle 0\left|\psi(\vec{x}_1)\psi(\vec{x}_2)\psi(\vec{x}_3)a^+_{p_1}a^+_{p_2}a^+_{p_3}\right|0\right\rangle$$

$$= \left\langle 0\left|\psi(\vec{x}_1)\psi(\vec{x}_2)\left[\delta^3\left(\vec{p}^{(3)} - \vec{p}_1\right) - a^+_{p_1}\psi_{\vec{p}}(\vec{x}_3)\right]a^+_{p_2}a^+_{p_3}\right|0\right\rangle$$

commuting in $p-$*space,*

$$= \left\langle 0\left|\psi(\vec{x}_1)\psi(\vec{x}_2)\left[\delta^3\left(\vec{p}^{(3)} - \vec{p}_1\right)a^+_{p_2}a^+_{p_3} \pm a^+_{p_1}\left[\delta^3\left(\vec{p}^{(3)} - \vec{p}_2\right)a^+_{p_3} \pm a^+_{p_2}\left[\delta^3\left(\vec{p}^{(3)} - \vec{p}_3\right)\right]\right]\right]\right|0\right\rangle$$

$$= \left\langle 0\right|\begin{bmatrix}\delta^3\left(\vec{p}^{(3)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(2)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(1)} - \vec{p}_1\right) - \delta^3\left(\vec{p}^{(3)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(1)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(2)} - \vec{p}_1\right)\\ +\delta^3\left(\vec{p}^{(2)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(1)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(3)} - \vec{p}_1\right) - \delta^3\left(\vec{p}^{(2)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(3)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(1)} - \vec{p}_1\right)\\ +\delta^3\left(\vec{p}^{(1)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(3)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(2)} - \vec{p}_1\right) - \delta^3\left(\vec{p}^{(1)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(2)} - \vec{p}_1\right)\delta^3\left(\vec{p}^{(3)} - \vec{p}_1\right)\end{bmatrix}\left|0\right\rangle$$

This clearly gives the Momentum-space wave functions (within a phase factor and normalization). Upon normalization and Fourier transformation back into real space, then, I will have exactly the functions obtained from the Slater determinant.

3) Consider a set of N uncoupled classical harmonic oscillators with

$$H = \sum_{i=1}^{N}\frac{P_i^2}{2} + \frac{1}{2}\omega_i^2 Q_i^2 \text{ .}$$

a) Find the classical equations of motion and show that a general solution is

$$Q_i = \frac{1}{\sqrt{2}}\left(\alpha_i e^{-i\omega_i t} + \alpha_i^* e^{i\omega_i t}\right)\text{.}$$

Why the complex conjugate?

$$\frac{\partial H}{\partial P_i} = \dot{Q}_i \qquad \frac{\partial H}{\partial Q_i} = -\dot{P}_i$$

Now verifying this general solution, I have:

$$\frac{\partial H}{\partial Q_i} = -\dot{P}_i$$

$$\omega_i^2 Q_i = \frac{\omega_i^{\ 2}}{\sqrt{2}}\left(\alpha_i e^{-i\omega_i t} + \alpha_i^* e^{i\omega_i t}\right) = -\dot{P}_i$$

$$\therefore \dot{P}_i = -\frac{\omega_i^{\ 2}}{\sqrt{2}}\left(\alpha_i e^{-i\omega_i t} + \alpha_i^* e^{i\omega_i t}\right)$$

$$\therefore P_i = -\frac{\omega_i^{\ 2}}{\sqrt{2}}\left(\frac{1}{-i\omega_i}\alpha_i e^{-i\omega_i t} + \frac{1}{i\omega_i}\alpha_i^* e^{i\omega_i t}\right) = \frac{\omega_i}{\sqrt{2}}\left(-i\alpha_i e^{-i\omega_i t} + i\alpha_i^* e^{i\omega_i t}\right)$$

Further,

$$\frac{\partial H}{\partial P_i} = \dot{Q}_i$$

$$P_i = \dot{Q}_i$$

$$\frac{\omega_i}{\sqrt{2}}\left(-i\alpha_i e^{-i\omega_i t} + i\alpha_i^* e^{i\omega_i t}\right) = \frac{1}{\sqrt{2}}\left(-i\omega_i\alpha_i e^{-i\omega_i t} + i\omega_i\alpha_i^* e^{i\omega_i t}\right)$$

as expected. Thus, I have used Hamilton's equations of motion to verify the validity of this particular general solution. The complex conjugation ensures a real result for position.

b) Use this solution to show that $H = \sum_i \omega_i^2\left(\alpha_i^*\alpha_i + \alpha_i\alpha_i^*\right)$.

$$H = \sum_{i=1}^{N}\frac{P_i^2}{2} + \frac{1}{2}\omega_i^2 Q_i^2 = \sum_{i=1}^{N}\frac{1}{2}\left(\frac{\omega_i}{\sqrt{2}}\left(-i\alpha_i e^{-i\omega_i t} + i\alpha_i^* e^{i\omega_i t}\right)\right)^2 + \frac{1}{2}\omega_i^2\left(\frac{1}{\sqrt{2}}\left(\alpha_i e^{-i\omega_i t} + \alpha_i^* e^{i\omega_i t}\right)\right)^2$$

$$= \sum_{i=1}^{N}\frac{\omega_i^{\ 2}}{4}\left(\left(-i\alpha_i e^{-i\omega_i t} + i\alpha_i^* e^{i\omega_i t}\right)^2 + \left(\alpha_i e^{-i\omega_i t} + \alpha_i^* e^{i\omega_i t}\right)^2\right)$$

$$= \sum_{i=1}^{N}\frac{\omega_i^{\ 2}}{4}\left(-\alpha_i^{\ 2} e^{-2i\omega_i t} + \alpha_i\alpha_i^* + \alpha_i^*\alpha_i - \alpha_i^{*2} e^{2i\omega_i t} + \alpha_i^{\ 2} e^{-2i\omega_i t} + \alpha_i\alpha_i^* + \alpha_i^*\alpha_i + \alpha_i^{*2} e^{2i\omega_i t}\right)$$

$$= \sum_{i=1}^{N}\frac{\omega_i^{\ 2}}{4}\left(2\alpha_i\alpha_i^* + 2\alpha_i^*\alpha_i\right) = \sum_{i=1}^{N}\frac{\omega_i^{\ 2}}{2}\left(\alpha_i\alpha_i^* + \alpha_i^*\alpha_i\right)$$

Note: This does not agree with the suggested solution.

c) Quantize by imposing canonical commutation relations $\left[P_i, Q_j\right] = -i\hbar\delta_{ij} \quad \left[P_i, P_j\right] = \left[Q_i, Q_j\right] = 0$. What are the commutation relations for α_i^+, α_j? Redefine the alphas so that $\alpha_i, \alpha_j = \delta_{ij}$ and write down the quantum Hamiltonian in terms of α^+, α.

$$[P_i, Q_j] = -i\hbar\delta_{ij}$$

$$\left[\frac{\omega_i}{\sqrt{2}}\left(-i\alpha_i e^{-i\omega_i t} + i\alpha_i^* e^{i\omega_i t}\right), \frac{1}{\sqrt{2}}\left(\alpha_j e^{-i\omega_j t} + \alpha_j^* e^{i\omega_j t}\right)\right] = -i\hbar\delta_{ij}$$

$$\frac{\omega_i}{2}\left(\left[-i\alpha_i e^{-i\omega_i t}, \alpha_i^* e^{i\omega_j t}\right] + \left[i\alpha_i^* e^{i\omega_i t}, \alpha_j e^{-i\omega_j t}\right]\right) = -i\hbar\delta_{ij}$$

$$\frac{i\omega_i}{2}\left(-\left[\alpha_i e^{-i\omega_i t}, \alpha_j^* e^{i\omega_j t}\right] + \left[\alpha_i^* e^{i\omega_i t}, \alpha_j e^{-i\omega_j t}\right]\right) = -i\hbar\delta_{ij}$$

$$Clearly, \quad [\alpha_i, \alpha_j^*] \propto \delta_{ij}$$

$$\frac{i\omega_i}{2}\left(-\left[\alpha_i e^{-i\omega_i t}, \alpha_i^* e^{i\omega_i t}\right] + \left[\alpha_i^* e^{i\omega_i t}, \alpha_i e^{-i\omega_i t}\right]\right) = \frac{i\omega_i}{2}\left(-[\alpha_i, \alpha_i^*] + [\alpha_i^*, \alpha_i]\right) = i\omega_i[\alpha_i^*, \alpha_i] = -i\hbar$$

$$\therefore [\alpha_i^*, \alpha_j] = -\frac{\hbar}{\omega_i}\delta_{ij}$$

$$re-define: \quad \alpha_i \to \sqrt{\frac{\hbar}{\omega_i}}\alpha'_i, \alpha_i^* \to \sqrt{\frac{\hbar}{\omega_i}}\alpha'^*_i$$

$$for \quad [\alpha'_i, \alpha'^*_j] = \delta_{ij}$$

The Quantum Hamiltonian is then

$$H = \sum_i \frac{\omega_i^2}{2}\left(\alpha_i^*\alpha_i + \alpha_i\alpha_i^*\right) = \sum_i \frac{\hbar\omega_i}{2}\left(\alpha'^*_i\,\alpha'_i + \alpha'_i\,\alpha'^*_i\right) = \sum_i \frac{\hbar\omega_i}{2}\left(2\alpha'^*_i\,\alpha'_i + 1\right) = \sum_i \hbar\omega_i\left(\alpha'^*_i\,\alpha'_i + \frac{1}{2}\right)$$

I have used the one I derived, rather than the one suggested from part (b). This gives the expected result.

4) **The Heisenberg equations of motion in second quantization for operators are** $i\hbar\frac{d\hat{\theta}(\vec{x},t)}{dt} = -[\hat{H}, \hat{\theta}(\vec{x},t)]$ **with** $\hat{\theta}(\vec{x},t)$ **an operator in the Heisenberg picture for** $\hat{H}(\vec{x}) = \int d^3x\,\psi^+(\vec{x})H(\vec{x})\psi(\vec{x})$**. Obtain the equations of motion for** $\psi(\vec{x}), \psi^+(\vec{x})$ **using the commutation relations for field operators.**

Thanks to CH for showing me the folly of my ways on this problem.

In the Heisenberg picture, let H be time-independent. Further, let

$$\hat{\theta}(\vec{x},t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{\theta}_S(\vec{x})e^{-\frac{i}{\hbar}\hat{H}t}$$

First, note that:

$$\left[\hat{H}, e^{\frac{i}{\hbar}\hat{H}t}\right] = 0 \quad because \quad [\hat{H}, \hat{H}] = 0$$

$$\therefore [\hat{H}, \hat{\theta}(\vec{x},t)] = e^{\frac{i}{\hbar}\hat{H}t}[\hat{H}, \theta_S(\vec{x})]e^{-\frac{i}{\hbar}\hat{H}t}$$

Now, let $\theta_S(\vec{x}) = \psi(\vec{x})$:

$$i\hbar \frac{d\psi(\vec{x},t)}{dt} = -e^{\frac{i}{\hbar}\hat{H}t}\left[\hat{H},\psi(\vec{x})\right]e^{-\frac{i}{\hbar}\hat{H}t}$$
$$\left[\hat{H},\psi(\vec{x})\right] = \int d^3x'\psi^+(\vec{x}')H(\vec{x})\psi(\vec{x}')\psi(\vec{x}) - \psi(\vec{x})\int d^3x'\psi^+(\vec{x}')H(\vec{x})\psi(\vec{x}')$$
$$use: \left[\psi(\vec{x}),\psi^+(\vec{x}')\right] = \delta(\vec{x}-\vec{x}')$$
$$= \int d^3x'\psi^+(\vec{x}')H(\vec{x}')\psi(\vec{x}')\psi(\vec{x}) - \psi(\vec{x})\int d^3x'\psi^+(\vec{x}')H(\vec{x})\psi(\vec{x}')$$
$$= \int d^3x'\psi^+(\vec{x}')\psi(\vec{x})H(\vec{x}')\psi(\vec{x}') - \psi(\vec{x})\int d^3x'\psi^+(\vec{x}')H(\vec{x})\psi(\vec{x}')$$
$$= \int d^3x'\left[-\delta(\vec{x}-\vec{x}') + \psi(\vec{x})\psi^+(\vec{x}')\right]H(\vec{x}')\psi(\vec{x}') - \psi(\vec{x})\int d^3x'\psi^+(\vec{x}')H(\vec{x})\psi(\vec{x}')$$
$$= -\int d^3x'\delta(\vec{x}-\vec{x}')H(\vec{x}')\psi(\vec{x}')$$
$$= -H(\vec{x})\psi(\vec{x})$$
$$\therefore i\hbar \frac{d\psi(\vec{x},t)}{dt} = e^{\frac{i}{\hbar}\hat{H}t}H(\vec{x})\psi(\vec{x})e^{-\frac{i}{\hbar}\hat{H}t}$$
$$\therefore i\hbar \frac{d\psi(\vec{x})}{dt} = H(\vec{x})\psi(\vec{x})$$

Analogously,

$$i\hbar \frac{d\psi^+(\vec{x},t)}{dt} = -e^{\frac{i}{\hbar}\hat{H}t}H(\vec{x})\psi^+(\vec{x})e^{-\frac{i}{\hbar}\hat{H}t}$$

5) **Consider the density operator $\hat{\rho}(\vec{x}) = \psi^+(\vec{x})\psi(\vec{x})$, using the commutation relations show that**
$$\hat{\rho}(\vec{y})\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)\ldots\psi^+(\vec{x}_N)\left|0\right\rangle = \sum_{i=1}^{N}\delta^3(\vec{y}-\vec{x}_i)\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)\ldots\psi^+(\vec{x}_N)\left|0\right\rangle.$$

$$\pm\begin{matrix} Bose \\ Fermi \end{matrix}$$
$$\hat{\rho}(\vec{y})\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)\ldots\psi^+(\vec{x}_N)\left|0\right\rangle$$
$$= \psi^+(\vec{y})\psi(\vec{y})\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)\ldots\psi^+(\vec{x}_N)\left|0\right\rangle$$
$$= \psi^+(\vec{y})\left[\delta^3(\vec{y}-\vec{x}_1) \pm \psi^+(\vec{x}_1)\psi(\vec{y})\right]\psi^+(\vec{x}_2)\ldots\psi^+(\vec{x}_N)\left|0\right\rangle$$
$$= \psi^+(\vec{y})\left[\delta^3(\vec{y}-\vec{x}_1)\psi^+(\vec{x}_2) \pm \psi^+(\vec{x}_1)\left[\delta^3(\vec{y}-\vec{x}_2) \pm \psi^+(\vec{x}_2)\psi(\vec{y})\right]\right]\psi^+(\vec{x}_3)\ldots\psi^+(\vec{x}_N)\left|0\right\rangle$$
$$after \quad \psi(\vec{y})\left|0\right\rangle \to 0:$$
$$= \psi^+(\vec{y})\sum_{i=1}^{N}\delta^3(\vec{y}-\vec{x}_i)\prod_{j=1,\, j\neq i}^{N}(\pm)^{j-1}\psi^+(\vec{x}_j)\left|0\right\rangle$$

After the integration (not shown here), of course, $\psi^+(\vec{y}) \to \psi^+(\vec{x}_i)$. It doesn't make a difference, however, if I make that identification now:

$$= \sum_{i=1}^{N} \delta^3(\vec{y} - \vec{x}_i)\psi^+(\vec{x}_i) \prod_{j=1,\, j\neq i}^{N} (\pm)^{i-1} \psi^+(\vec{x}_j)|0\rangle$$

Now consider that in order to obtain normal ordering, the $\psi^+(\vec{x}_i)$ will pass through i – 1 commutations, each bringing a factor of ± appropriate to the statistics for this particular particle. Then, I will have:

$$= \sum_{i=1}^{N} \delta^3(\vec{y} - \vec{x}_i) \prod_{j=1}^{N} (\pm)^{i-1} (\pm)^{i-1} \psi^+(\vec{x}_j)|0\rangle = \sum_{i=1}^{N} \delta^3(\vec{y} - \vec{x}_i)\psi^+(\vec{x}_1)\psi^+(\vec{x}_2)\ldots\psi^+(\vec{x}_N)|0\rangle$$

just as expected.