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Quantum Mechanics 3 Homework 11

1) **<u>Weakly interacting Bose gas in the Bogoliubov approximation</u>:**

$$\hat{H} = \sum_k \varepsilon(k)\hat{a}_k^+\hat{a}_k + \frac{1}{2V}\sum_q V_q \sum_{p,k} \hat{a}_{p+q}^+\hat{a}_{k-q}^+\hat{a}_k\hat{a}_p$$

Consider $\hat{K} = \hat{H} - \mu\hat{N}; \hat{N} = \sum_k \hat{a}_k^+\hat{a}_k$ **.**

Define: $\hat{a}_0 = \sqrt{N_0}e^{i\theta} + \hat{b}_0, \hat{a}_{k\neq 0} = \hat{b}_{k\neq 0}$

a) **Separate the terms of order** $N_0^2, N_0\sqrt{N_0}, N_0$ **in the interaction term, show that these are quadratic in** $\hat{b},\hat{b}^+$**, and show that terms of order** $\sqrt{N_0}$ **and** 1 **are cubic and quartic in** $\hat{b},\hat{b}^+$**. Neglect the terms of** $\theta\left(\sqrt{N_0}\right),\theta(1)$ **(Bogoliubov approximation) and write down** $\hat{K}$ **keeping these terms only (up to quadratic order in** $\hat{b},\hat{b}^+$**.**

The first term is broken up into two cases: $k = 0, k \neq 0$.

Further, I take in this case $\varepsilon(0) = 0$

The second term is broken up into nine terms corresponding to each permutation of: $\{p = -q; p = 0, q \neq 0; p \neq -q, p \neq 0\}$ and $\{k = q; k = 0, q \neq 0; k \neq q, k \neq 0\}$. Several of these terms corresponding to the first options in the lists above will require further consideration where $q = 0$ might generate more zero-indexed terms. Let me show these nine permutations first:

$$\hat{H} = \sum_k \varepsilon(k)\hat{a}_k^+\hat{a}_k + \frac{1}{2V}\sum_q V_q \sum_{p,k} \hat{a}_{p+q}^+\hat{a}_{k-q}^+\hat{a}_k\hat{a}_p$$
$$= \sum_{k\neq 0} \varepsilon(k)\hat{a}_k^+\hat{a}_k$$
$$+ \frac{1}{2V}\sum_q V_q \hat{a}_0^+\hat{a}_0^+\hat{a}_q\hat{a}_{-q} + \frac{1}{2V}\sum_{q\neq 0} V_q \hat{a}_0^+\hat{a}_{k-q}^+\hat{a}_0\hat{a}_{-q} + \frac{1}{2V}\sum_q V_q \sum_{k\neq q,k\neq 0} \hat{a}_0^+\hat{a}_{k-q}^+\hat{a}_k\hat{a}_{-q}$$
$$+ \frac{1}{2V}\sum_{q\neq 0} V_q \hat{a}_q^+\hat{a}_0^+\hat{a}_q\hat{a}_0 + \frac{1}{2V}\sum_{q\neq 0} V_q \hat{a}_q^+\hat{a}_{-q}^+\hat{a}_0\hat{a}_0 + \frac{1}{2V}\sum_{q\neq 0} V_q \sum_{k\neq q,k\neq 0} \hat{a}_q^+\hat{a}_{k-q}^+\hat{a}_k\hat{a}_0$$
$$+ \frac{1}{2V}\sum_q V_q \hat{a}_{p+q}^+\hat{a}_0^+\hat{a}_q\hat{a}_p + \frac{1}{2V}\sum_{q\neq 0} V_q \hat{a}_{p+q}^+\hat{a}_{-q}^+\hat{a}_0\hat{a}_p + \frac{1}{2V}\sum_q V_q \sum_{p\neq -q, p\neq 0, k\neq q, k\neq 0} \hat{a}_{p+q}^+\hat{a}_{k-q}^+\hat{a}_k\hat{a}_p$$

Splitting these, then, where necessary into $q = 0$ and $q \neq 0$ cases, I have:

$$\hat{H} = (one - particle \quad energies): \sum_{k\neq 0} \varepsilon(k) \hat{a}_k^+ \hat{a}_k$$

$$(0 \quad excited \quad in.teraction): + \frac{1}{2V} V_0 \hat{a}_0^+ \hat{a}_0^+ \hat{a}_0 \hat{a}_0$$

$$(2 \quad excited \quad in.teraction): + \frac{1}{2V} \sum_{q\neq 0} V_q \hat{a}_0^+ \hat{a}_0^+ \hat{a}_q \hat{a}_{-q} + \frac{1}{2V} \sum_{q\neq 0} V_q \hat{a}_0^+ \hat{a}_{-q}^+ \hat{a}_0 \hat{a}_{-q} + \frac{1}{2V} V_0 \sum_{k\neq 0} \hat{a}_0^+ \hat{a}_k^+ \hat{a}_k \hat{a}_0$$

$$+ \frac{1}{2V} \sum_{q\neq 0} V_q \hat{a}_q^+ \hat{a}_0^+ \hat{a}_q \hat{a}_0 + \frac{1}{2V} \sum_{q\neq 0} V_q \hat{a}_q^+ \hat{a}_{-q}^+ \hat{a}_0 \hat{a}_0 + \frac{1}{2V} V_0 \sum_{p\neq 0} \hat{a}_p^+ \hat{a}_0^+ \hat{a}_0 \hat{a}_p$$

$$(3 \quad excited \quad in.teraction): + \frac{1}{2V} \sum_{q\neq 0} V_q \sum_{k\neq q, k\neq 0} \hat{a}_0^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_{-q} + \frac{1}{2V} \sum_{q\neq 0} V_q \sum_{k\neq q, k\neq 0} \hat{a}_q^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_0$$

$$+ \frac{1}{2V} \sum_{q\neq 0, p\neq -q, p\neq 0} V_q \hat{a}_{p+q}^+ \hat{a}_0^+ \hat{a}_q \hat{a}_p + \frac{1}{2V} \sum_{q\neq 0, p\neq -q, p\neq 0} V_q \hat{a}_{p+q}^+ \hat{a}_{-q}^+ \hat{a}_0 \hat{a}_p$$

$$(4 \quad excited \quad in.teraction): + \frac{1}{2V} \sum_q V_q \sum_{p\neq -q, p\neq 0, k\neq q, k\neq 0} \hat{a}_{p+q}^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_p$$

The Feynman diagrams of these show the various processes of scattering, including pair production and creation from the condensate. Now switching over to the $\hat{b}$ language, and considering only the interaction terms:

$$\hat{H}_I =$$

$$(0 \quad excited \quad in.teraction): + \frac{1}{2V} V_0 \left(\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+\right)\left(\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+\right)\left(\sqrt{N_0} e^{i\theta} + \hat{b}_0\right)\left(\sqrt{N_0} e^{i\theta} + \hat{b}_0\right)$$

$$(2 \quad excited \quad in.teraction): + \frac{1}{2V} \sum_{q\neq 0} V_q \left(\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+\right)\left(\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+\right)\hat{b}_q \hat{b}_{-q}$$

$$+ \frac{1}{2V} \sum_{q\neq 0} V_q \left(\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+\right)\hat{b}_{-q}^+ \left(\sqrt{N_0} e^{i\theta} + \hat{b}_0\right)\hat{b}_{-q} + \frac{1}{2V} V_0 \sum_{k\neq 0} \left(\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+\right)\hat{b}_k^+ \hat{b}_k \left(\sqrt{N_0} e^{i\theta} + \hat{b}_0\right)$$

$$+ \frac{1}{2V} \sum_{q\neq 0} V_q \hat{b}_q^+ \left(\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+\right)\hat{b}_q \left(\sqrt{N_0} e^{i\theta} + \hat{b}_0\right) + \frac{1}{2V} \sum_{q\neq 0} V_q \hat{b}_q^+ \hat{b}_{-q}^+ \left(\sqrt{N_0} e^{i\theta} + \hat{b}_0\right)\left(\sqrt{N_0} e^{i\theta} + \hat{b}_0\right)$$

$$+ \frac{1}{2V} V_0 \sum_{p\neq -q, p\neq 0} \hat{b}_p^+ \left(\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+\right)\left(\sqrt{N_0} e^{i\theta} + \hat{b}_0\right)\hat{b}_p$$

$$(3 \quad excited \quad in.teraction): + \frac{1}{2V} \sum_{q\neq 0} V_q \sum_{k\neq q, k\neq 0} \left(\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+\right)\hat{b}_{k-q}^+ \hat{b}_k \hat{b}_{-q} + \frac{1}{2V} \sum_{q\neq 0} V_q \sum_{k\neq q, k\neq 0} \hat{b}_q^+ \hat{b}_{k-q}^+ \hat{b}_k \left(\sqrt{N_0} e^{i\theta} + \hat{b}_0\right)$$

$$+ \frac{1}{2V} \sum_{q\neq 0} V_q \sum_{p\neq -q, p\neq 0} \hat{b}_{p+q}^+ \left(\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+\right)\hat{b}_q \hat{b}_p + \frac{1}{2V} \sum_{q\neq 0} V_q \sum_{p\neq -q, p\neq 0} \hat{b}_{p+q}^+ \hat{b}_{-q}^+ \left(\sqrt{N_0} e^{i\theta} + \hat{b}_0\right)\hat{b}_p$$

$$(4 \quad excited \quad in.teraction): + \frac{1}{2V} \sum_q V_q \sum_{p\neq -q, p\neq 0, k\neq q, k\neq 0} \hat{b}_{p+q}^+ \hat{b}_{k-q}^+ \hat{b}_k \hat{b}_p$$

Rather than explicitly expanding these in all of their terrible glory, I note a pattern: the only way to pick up a factor of $\sqrt{N_0}$ in the above is to sacrifice the presence of a $\hat{b}_0$. In this way, then, I see that for each factor of $\sqrt{N_0}$ that appears, the order of the term is reduced by one. Then, terms up to quadratic in $\hat{b}_0$ have coefficients $N_0^2, N_0\sqrt{N_0}, N_0$ and higher order terms have coefficients $\sqrt{N_0}, 1$. Then, dropping the latter-order terms, I get:

$$
\begin{aligned}
&(0 \quad excited \quad in.teraction) \rightarrow \\
&V_0\left[N_0^2 + 2N_0\sqrt{N_0}e^{i\theta}\hat{b}_0^+ + N_0 e^{2i\theta}\hat{b}_0^+\hat{b}_0^+ + 2N_0\sqrt{N_0}e^{-i\theta}\hat{b}_0 + 4N_0\hat{b}_0^+\hat{b}_0 + N_0 e^{-2i\theta}\hat{b}_0\hat{b}_0 + \theta\left(\sqrt{N_0}\right) + \theta(1)\right] \\
&(2 \quad excited \quad in.teraction) \rightarrow \\
&+\frac{1}{2V}\sum_{q\neq 0} V_q N_0 e^{-2i\theta}\hat{b}_q\hat{b}_{-q} + \frac{1}{2V}\sum_{q\neq 0} V_q N_0\hat{b}_{-q}^+\hat{b}_{-q} + \frac{1}{2V}V_0\sum_{k\neq 0} N_0\hat{b}_k^+\hat{b}_k \\
&+\frac{1}{2V}\sum_{q\neq 0} V_q N_0\hat{b}_q^+\hat{b}_q + \frac{1}{2V}\sum_{q\neq 0} V_q N_0 e^{2i\theta}\hat{b}_q^+\hat{b}_{-q}^+ + \frac{1}{2V}V_0\sum_{p\neq 0} N_0\hat{b}_p^+\hat{b}_p + \theta\left(\sqrt{N_0}\right) \\
&(3 \quad excited \quad in.teraction) \rightarrow \\
&+\theta\left(\sqrt{N_0}\right) + \theta(1) \\
&(4 \quad excited \quad in.teraction) :\rightarrow \\
&+\theta(1)
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\hat{K} = \hat{H} - \mu\hat{N} \\
&= \sum_{k\neq 0}\varepsilon(k)\hat{b}_k^+\hat{b}_k \\
&+\frac{V_0}{2V}N_0^2 + 2\frac{V_0}{2V}N_0\sqrt{N_0}e^{i\theta}\hat{b}_0^+ + \frac{V_0}{2V}N_0 e^{2i\theta}\hat{b}_0^+\hat{b}_0^+ + 2\frac{V_0}{2V}N_0\sqrt{N_0}e^{-i\theta}\hat{b}_0 + 4\frac{V_0}{2V}N_0\hat{b}_0^+\hat{b}_0 + \frac{V_0}{2V}N_0 e^{-2i\theta}\hat{b}_0\hat{b}_0 \\
&+\frac{1}{2V}\sum_{q\neq 0} V_q N_0 e^{-2i\theta}\hat{b}_q\hat{b}_{-q} + \frac{1}{2V}\sum_{q\neq 0} V_q N_0\hat{b}_{-q}^+\hat{b}_{-q} + \frac{1}{2V}V_0\sum_{k\neq 0} N_0\hat{b}_k^+\hat{b}_k \\
&+\frac{1}{2V}\sum_{q\neq 0} V_q N_0\hat{b}_q^+\hat{b}_q + \frac{1}{2V}\sum_{q\neq 0} V_q N_0 e^{2i\theta}\hat{b}_q^+\hat{b}_{-q}^+ + \frac{1}{2V}V_0\sum_{p\neq 0} N_0\hat{b}_p^+\hat{b}_p - \mu\hat{N} + \theta\left(\sqrt{N_0}\right) + \theta(1)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k}\varepsilon(k)\hat{b}_k^+\hat{b}_k \\
&+\frac{V_0}{V}N_0\sqrt{N_0}e^{i\theta}\hat{b}_0^+ + \frac{V_0}{V}N_0\sqrt{N_0}e^{-i\theta}\hat{b}_0 + \frac{1}{2V}\sum_{q} V_q N_0 e^{-2i\theta}\hat{b}_q\hat{b}_{-q} + \frac{2}{V}\sum_{q} V_q N_0\hat{b}_{-q}^+\hat{b}_{-q} \\
&+\frac{1}{2V}\sum_{q} V_q N_0 e^{2i\theta}\hat{b}_q^+\hat{b}_{-q}^+ + \frac{V_0}{2V}N_0^2 - \mu\hat{N} + \theta\left(\sqrt{N_0}\right) + \theta(1)
\end{aligned}
$$

b) Show that in this Bogoliubov approximation that $\hat{K} = \hat{K}_Q + \hat{K}_{cl}$ where $\hat{K}_Q$ is quadratic and linear in $\hat{b}, \hat{b}^+$ and $\hat{K}_{cl}$ is purely classical and independent of $\hat{b}, \hat{b}^+$. Establish a relation $\mu = \mu(N_0)$ by minimization of $\hat{K}_{cl}$: $\left.\frac{\partial \hat{K}_{cl}}{\partial N_0}\right|_{\mu} = 0$.

This is the Gross-Ginzburg-Pitaevskii equation. Show that imposing this condition leads to the cancellation of the terms linear in $\hat{b}, \hat{b}^+$ in $\hat{K}_Q$.

$$\mu\hat{N} = \mu\left(\sqrt{N_0}e^{-i\theta} + \hat{b}_0^+\right)\left(\sqrt{N_0}e^{i\theta} + \hat{b}_0\right) + \mu\sum_k \hat{b}_k^+\hat{b}_k$$

then:

$$\hat{K} = \hat{K}_Q + \hat{K}_{cl}$$

$$\hat{K}_Q = \sum_k \varepsilon(k)\hat{b}_k^+\hat{b}_k$$

$$+\sqrt{N_0}\left[\frac{V_0}{V}N_0 - \mu\right]e^{i\theta}\hat{b}_0^+ + \frac{V_0}{V}\sqrt{N_0}\left[\frac{V_0}{V}N_0 - \mu\right]e^{-i\theta}\hat{b}_0 + \frac{1}{2V}\sum_q V_q N_0 e^{-2i\theta}\hat{b}_q\hat{b}_{-q} + \frac{2}{V}\sum_q V_q N_0 \hat{b}_{-q}^+\hat{b}_{-q}$$

$$+\frac{1}{2V}\sum_q V_q N_0 e^{2i\theta}\hat{b}_q^+\hat{b}_{-q}^+ - \mu\sum_k \hat{b}_k^+\hat{b}_k + \theta\left(\sqrt{N_0}\right) + \theta(1)$$

$$\hat{K}_{cl} = \frac{V_0}{2V}N_0^2 - \mu N_0$$

Classically, then,

$$\hat{K}_{cl} \to \frac{V_0}{2V}N_0^2 - \mu N_0$$

$$\frac{\partial \hat{K}_{cl}}{\partial N_0} = \frac{V_0}{V}N_0 - \mu$$

$$\frac{\partial \hat{K}_{cl}}{\partial N_0} = 0 \to \mu = \frac{V_0}{V}N_0$$

Under this substitution, then:

$$\hat{K}_Q = \sum_k \varepsilon(k)\hat{b}_k^+\hat{b}_k$$

$$+\frac{1}{2V}\sum_q V_q N_0 e^{-2i\theta}\hat{b}_q\hat{b}_{-q} + \frac{2}{V}\sum_q V_q N_0 \hat{b}_{-q}^+\hat{b}_{-q}$$

$$+\frac{1}{2V}\sum_q V_q N_0 e^{2i\theta}\hat{b}_q^+\hat{b}_{-q}^+ - \mu\sum_k \hat{b}_k^+\hat{b}_k + \theta\left(\sqrt{N_0}\right) + \theta(1)$$

The terms related to $\varepsilon(0)$ and μ now correspond to a zero-point energy and can be dropped, leaving only terms quadratic in $\hat{b},\hat{b}^{+}$:

$$\hat{K}_Q = \sum_k \varepsilon(k)\hat{b}_k^{+}\hat{b}_k + \frac{1}{2V}\sum_q V_q N_0 e^{-2i\theta}\hat{b}_q\hat{b}_{-q} + \frac{2}{V}\sum_q V_q N_0 \hat{b}_{-q}^{+}\hat{b}_{-q}$$

$$+\frac{1}{2V}\sum_q V_q N_0 e^{2i\theta}\hat{b}_q^{+}\hat{b}_{-q}^{+} + \theta\left(\sqrt{N_0}\right) + \theta(1)$$

c) **Diagonalize the resulting quadratic form for $\hat{K}_Q$ by a Bogoliubov transformation:**

$$\hat{b}_k^{+}e^{i\theta} = \hat{c}_k \cosh\phi_k + \hat{c}_{-k}^{+}\sinh\phi_k \textbf{.}$$

Find $\cosh\phi_k, \sinh\phi_k$ **by requesting the cancellation of terms** $\hat{c}_k^{+}\hat{c}_{-k}^{+}, \hat{c}_k\hat{c}_{-k}$ **. Show that in this Bogoliubov transformation,**

$$\hat{K}_Q = \sum_k \hat{c}_k^{+}\hat{c}_k\hbar\Omega(k) + K_0 .$$

Find $\Omega(k)$ **and** K_0**, consider** $V_q \equiv V_0$ **constant and evaluate the integral for** K_0 **.**

Let ϕ_k, V_q and $\varepsilon(k)$ be even functions of k.
Let $\left[\hat{c}_k^{+},\hat{c}_k\right]=1$, as shown in the assignment 10.

$$\hat{K}_Q = \sum_k \left(\varepsilon(k) + \frac{2V_k N_0}{V}\right)\left(\hat{c}_k\cosh\phi_k + \hat{c}_{-k}^{+}\sinh\phi_k\right)\left(\hat{c}_k^{+}\cosh\phi_k + \hat{c}_{-k}\sinh\phi_k\right)$$

$$+\frac{1}{2V}\sum_k V_k N_0\left(\hat{c}_k^{+}\cosh\phi_k + \hat{c}_{-k}\sinh\phi_k\right)\left(\hat{c}_{-k}^{+}\cosh\phi_{-k} + \hat{c}_k\sinh\phi_k\right)$$

$$+\frac{1}{2V}\sum_k V_k N_0\left(\hat{c}_k\cosh\phi_k + \hat{c}_{-k}^{+}\sinh\phi_k\right)\left(\hat{c}_{-k}\cosh\phi_k + \hat{c}_k^{+}\sinh\phi_k\right)$$

Now I have, after commuting and eliminating terms as well as re-indexing where necessary that

$$\hat{K}_Q = \sum_k \left(\left(\varepsilon(k) + \frac{2V_k N_0}{V} \right) \left(\sinh^2 \phi_k + \cosh^2 \phi_k \right) + \frac{2V_k N_0}{V} \sinh \phi_k \cosh \phi_k \right) \hat{c}_k^+ \hat{c}_k$$
$$+ \sum_k \left[\frac{V_k N_0}{2V} \left(\cosh^2 \phi_k + \sinh^2 \phi_k \right) + \left(\varepsilon(k) + \frac{2V_k N_0}{V} \right) \cosh \phi_k \sinh \phi_k \right] \hat{c}_k^+ \hat{c}_{-k}^+$$
$$+ \sum_k \left[\frac{V_k N_0}{2V} \left(\cosh^2 \phi_k + \sinh^2 \phi_k \right) + \left(\varepsilon(k) + \frac{2V_k N_0}{V} \right) \cosh \phi_k \sinh \phi_k \right] \hat{c}_k \hat{c}_{-k}$$
$$+ \frac{1}{V} \sum_k V_k N_0 \sinh \phi_k \cosh \phi_k - \sum_k \left(\varepsilon(k) + \frac{2V_k N_0}{V} \right) \cosh^2 \phi_k$$

This requires that

$$\frac{V_k N_0}{2V} \left(\cosh^2 \phi_k + \sinh^2 \phi_k \right) + \left(\varepsilon(k) + \frac{2V_k N_0}{V} \right) \cosh \phi_k \sinh \phi_k = 0$$
$$\frac{V_k N_0}{2V} \cosh 2\phi_k + \frac{1}{2} \left(\varepsilon(k) + \frac{2V_k N_0}{V} \right) \sinh 2\phi_k = 0 \quad ,$$

and so that

$$\cosh\ 2\phi_k = \frac{\frac{\varepsilon(k)}{2} + \frac{V_k N_0}{V}}{\sqrt{\left(\frac{\varepsilon(k)}{2} + \frac{V_k N_0}{V} \right)^2 - \left(\frac{N_0 V_k}{2V} \right)^2}}$$
$$\sinh\ 2\phi_k = \frac{-\frac{V_k N_0}{2V}}{\sqrt{\left(\frac{\varepsilon(k)}{2} + \frac{V_k N_0}{V} \right)^2 - \left(\frac{N_0 V_k}{2V} \right)^2}} \quad .$$

Clearly satisfies the relation $\cosh^2 x - \sinh^2 x = 1$.

Now:

$$\hbar\Omega(k) = 2 \left(\left(\frac{\varepsilon(k)}{2} + \frac{V_k N_0}{V} \right) \cosh 2\phi_k + \frac{V_k N_0}{2V} \sinh 2\phi_k \right)$$
$$= 2 \sqrt{\left(\frac{\varepsilon(k)}{2} + \frac{V_k N_0}{V} \right)^2 - \left(\frac{N_0 V_k}{2V} \right)^2}$$

Also,

$$K_0 = \frac{1}{V}\sum_k V_k N_0 \sinh\phi_k \cosh\phi_k - \sum_k\left(\varepsilon(k)+\frac{2V_k N_0}{V}\right)\cosh^2\phi_k$$

$$= \frac{1}{2V}\sum_k V_k N_0 \sinh 2\phi_k - \frac{1}{2}\sum_k\left(\varepsilon(k)+\frac{2V_k N_0}{V}\right)(\cosh 2\phi_k + 1)$$

$$= \sum_k\sqrt{\left(\frac{\varepsilon(k)}{2}+\frac{V_k N_0}{V}\right)^2 - \left(\frac{N_0 V_k}{2V}\right)^2} - \sum_k\left(\frac{\varepsilon(k)}{2}+\frac{V_k N_0}{V}\right)$$

$$\to 4\pi\int_0^{k_F} k^2 dk\sqrt{\left(\frac{\hbar^2 k^2}{4m}+\frac{V_0 N_0}{V}\right)^2 - \left(\frac{N_0 V_0}{2V}\right)^2} - 4\pi\int_0^{k_F} k^2 dk\left(\frac{\hbar^2 k^2}{4m}+\frac{V_0 N_0}{V}\right)$$

Note that the answer correctly vanishes when N_0 vanishes.

2)

a) Invert the Bogoliubov transformation in item c in problem 1 and show that $\hat{c}_k = \tilde{b}_k \cosh\phi_k - \tilde{b}_{-k}^+ \sinh\phi_k ; \tilde{b}_k = \hat{b}_k e^{-i\theta}$. Use $e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \ldots$ to show that $\hat{c}_k = U(\phi)\tilde{b}_k U^{-1}(\phi)$ where $U(\phi)$ is the unitary operator $U(\phi) = e^{\sum_{k>0}\phi_k\left(\tilde{b}_k^+\tilde{b}_{-k}^+ - \tilde{b}_k\tilde{b}_{-k}\right)}$.

$$\tilde{b}_k^+ = \hat{b}_k^+ e^{i\theta} = \hat{c}_k \cosh\phi_k + \hat{c}_{-k}^+ \sinh\phi_k$$
$$\tilde{b}_k = \hat{b}_k e^{-i\theta} = \hat{c}_k^+ \cosh\phi_k + \hat{c}_{-k} \sinh\phi_k$$

then

$$\tilde{b}_k^+ \cosh\phi_k + \tilde{b}_{-k} \sinh\phi_k = \left[\hat{c}_k \cosh\phi_k + \hat{c}_{-k}^+ \sinh\phi_k\right]\cosh\phi_k - \left[\hat{c}_{-k}^+ \cosh\phi_k + \hat{c}_k \sinh\phi_k\right]\sinh\phi_k$$
$$= \hat{c}_k\left(\cosh^2\phi_k - \sinh^2\phi_k\right) = \hat{c}_k$$

Further, taking $U(\phi) = e^{\sum_{k>0}\phi_k\left(\tilde{b}_k^+\tilde{b}_{-k}^+ - \tilde{b}_k\tilde{b}_{-k}\right)}$, I have

$$U(\phi)\tilde{b}_k U^{-1}(\phi) = \tilde{b}_k + \left[\sum_{i>0}\phi_i\left(\tilde{b}_i^+\tilde{b}_{-i}^+ - \tilde{b}_i\tilde{b}_{-i}\right), \tilde{b}_k\right] + \frac{1}{2!}\left[\sum_{i>0}\phi_i\left(\tilde{b}_i^+\tilde{b}_{-i}^+ - \tilde{b}_i\tilde{b}_{-i}\right), \left[\sum_{j>0}\phi_j\left(\tilde{b}_j^+\tilde{b}_{-j}^+ - \tilde{b}_j\tilde{b}_{-j}\right), \tilde{b}_k\right]\right] + \ldots$$

Certainly, $\left[\hat{b}_i^+, \hat{b}_j\right] = \delta_{ij}$.

Now I have that

$$\left[\sum_{i>0}\phi_i\left(\tilde{b}_i^+\tilde{b}_{-i}^+ - \tilde{b}_i\tilde{b}_{-i}\right), \tilde{b}_k\right] = \phi_k \tilde{b}_{-k}^+$$

Using this result in the next element, I have:

$$\left[\sum_{i>0}\phi_i\left(\tilde{b}_i^+\tilde{b}_{-i}^+-\tilde{b}_i\tilde{b}_{-i}\right),\left[\sum_{j>0}\phi_j\left(\tilde{b}_j^+\tilde{b}_{-j}^+-\tilde{b}_j\tilde{b}_{-j}\right),\tilde{b}_k\right]\right]=\left[\sum_{i>0}\phi_i\left(\tilde{b}_i^+\tilde{b}_{-i}^+-\tilde{b}_i\tilde{b}_{-i}\right),\phi_k\tilde{b}_{-k}^+\right]=-\phi_k^2\tilde{b}_k$$

And now I notice a pattern. Namely, the N -commutator will have value:

$$N=\begin{cases} even: \phi_k{}^N\tilde{b}_k \\ odd: -\phi_k{}^N\tilde{b}_k^+ \end{cases}$$

Now I see that these entries correspond to the Taylor expansion of

$$\cosh\phi_k = 1+\frac{1}{2!}\phi_k^2+\frac{1}{4!}\phi_k^4+\dots$$

in the case of the coefficients multiplying $\tilde{b}_k$ and

$$\sin\phi_k = \phi_k+\frac{1}{3!}\phi_k^3+\frac{1}{5!}\phi_k^5+\dots$$

in the case of the terms multiplying $\tilde{b}_k^+$,
and so overall I have that

$$U(\phi)\tilde{b}_k U^{-1}(\phi)=\tilde{b}_k\cosh\phi_k-\tilde{b}_{-k}^+\sinh\phi_k=\hat{c}_k$$

just as expected.

b) Show that the ground state of $\hat{K}$ in the Bogoliubov approximation is $\left|GS\right\rangle=U(\phi)\left|\tilde{0}\right\rangle$ where $\left|\tilde{0}\right\rangle$ is the vacuum of the operators $\tilde{b}_k:\tilde{b}_k\left|\tilde{0}\right\rangle=0$ for all k. Argue that $\left|GS\right\rangle$ is a linear superposition of states with a pair of momenta $\vec{k},-\vec{k}$ respectively. This is a squeezed quantum state. These states are ubiquitous in quantum optics and quantum controlled nanoscale systems.

From part 1c, $\hat{K}_Q=\sum_k\hat{c}_k^+\hat{c}_k\hbar\Omega(k)+K_0$. Dropping the zero-point term, I have then

$$\hat{K}'_Q=\sum_k\hat{c}_k^+\hat{c}_k\hbar\Omega(k)=\sum_k\hbar\Omega(k)U(\phi)\tilde{b}_k^+U^{-1}(\phi)U(\phi)\tilde{b}_kU^{-1}(\phi)$$

Now using the result from 2a, I have:

$\hat{K}'_Q\,|GS\rangle$
$= \hbar\Omega(k)U(\phi)\tilde{b}_k^+U^{-1}(\phi)U(\phi)\tilde{b}_kU^{-1}(\phi)U(\phi)|\tilde{0}\rangle$
$= \hbar\Omega(k)U(\phi)\tilde{b}_k^+U^{-1}(\phi)U(\phi)\tilde{b}_k|\tilde{0}\rangle = 0$

Thus I see that $|GS\rangle$ is the ground state of the Bogoliubov approximation, since $\tilde{b}_k|\tilde{0}\rangle = 0$.

Consider now the action of $U(\phi) = e^{\sum_{k>0}\phi_k\left(\tilde{b}_k^+\tilde{b}_{-k}^+ - \tilde{b}_k\tilde{b}_{-k}\right)}$ on the vacuum $|\tilde{0}\rangle$. Note that from this form, I see that particles are only created in a pair $\tilde{b}_k^+\tilde{b}_{-k}^+$ and are only destroyed in a pair $\tilde{b}_k\tilde{b}_{-k}$. This observation alone is sufficient to ensure that all states that exist can only have equal contributions from particles of momentum $\vec{k},-\vec{k}$.

3) **Mean-Field Theory, Coherent States, and the Gross-Pitaevski Equation: Consider the pair potential** $V(\vec{x}-\vec{y}) = V_0\delta^3(\vec{x}-\vec{y})$ **and introduce the coherent states of the Bosonic operator** $|\psi(\vec{x})\rangle$ **such that** $\hat{\psi}(\vec{x})|\psi(\vec{x})\rangle = \psi(\vec{x})|\psi(\vec{x})\rangle$**. Include a one-body "trap" potential in the Hamiltonian** $\hat{H}$ **:**

$$\hat{H} = \int d^3x\hat{\psi}^+(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x})\right]\hat{\psi}(\vec{x}) + \frac{1}{2}\int d^3xd^3y\hat{\psi}^+(\vec{x})\hat{\psi}^+(\vec{y})V(\vec{x}-\vec{y})\hat{\psi}(\vec{y})\hat{\psi}(\vec{x})$$

.

a) **Minimize the energy** $E(\psi) = \langle\psi|\hat{K}|\psi\rangle$ **where** $|\psi\rangle$ **is the coherent state above** $\langle\psi|\psi\rangle = 1$ **and show that** $\frac{\partial E}{\partial\psi^*(\vec{x})} = 0$ **leads to the Gross-Pitaevskii equation** $\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu\right]\psi(\vec{x}) + V_0|\psi(\vec{x})|^2\psi(\vec{x}) = 0$ **with the constraint that** $\int|\psi(\vec{x})|^2d^3x = N$ **.**

$\hat{K} = \hat{H} - \mu\hat{N}$
$Take: \hat{N} = \int d^3x\hat{\psi}^+(\vec{x})\hat{\psi}(\vec{x})$
$= \hat{H} - \mu\int d^3x\hat{\psi}^+(\vec{x})\hat{\psi}(\vec{x})$

Then:

$$E(\psi) = \langle \psi | \hat{K} | \psi \rangle = 0$$

$$= \langle \psi | \int d^3x \hat{\psi}^+(\vec{x}) \left[-\frac{\hbar^2 \vec{\nabla}^2}{2m} + U(\vec{x}) \right] \hat{\psi}(\vec{x}) + \frac{1}{2} \int d^3x d^3y \hat{\psi}^+(\vec{x}) \hat{\psi}^+(\vec{y}) V(\vec{x}-\vec{y}) \hat{\psi}(\vec{y}) \hat{\psi}(\vec{x}) - \mu \int d^3x \hat{\psi}^+(\vec{x}) \hat{\psi}(\vec{x}) | \psi \rangle$$

$$= \int d^3x \psi^*(\vec{x}) \left[-\frac{\hbar^2 \vec{\nabla}^2}{2m} + U(\vec{x}) \right] \psi(\vec{x}) + \frac{1}{2} \int d^3x d^3y \psi^*(\vec{x}) \psi^*(\vec{y}) V(\vec{x}-\vec{y}) \psi(\vec{y}) \psi(\vec{x}) - \mu \int d^3x \psi^*(\vec{x}) \psi(\vec{x})$$

$$\frac{\partial E}{\partial \psi^*} = \int d^3x \left[-\frac{\hbar^2 \vec{\nabla}^2}{2m} + U(\vec{x}) \right] \psi(\vec{x}) + \frac{1}{2} \int d^3x d^3y \psi^*(\vec{y}) V(\vec{x}-\vec{y}) \psi(\vec{y}) \psi(\vec{x})$$

$$+ \frac{1}{2} \int d^3x d^3y \psi^*(\vec{x}) V(\vec{x}-\vec{y}) \psi(\vec{y}) \psi(\vec{x}) - \mu \int d^3x \psi(\vec{x})$$

$$= \int d^3x \left[-\frac{\hbar^2 \vec{\nabla}^2}{2m} + U(\vec{x}) - \mu \right] \psi(\vec{x}) + \int d^3x d^3y \psi^*(\vec{y}) V_0 \delta^3(\vec{x}-\vec{y}) \psi(\vec{y}) \psi(\vec{x})$$

$$= \int d^3x \left[-\frac{\hbar^2 \vec{\nabla}^2}{2m} + U(\vec{x}) - \mu \right] \psi(\vec{x}) + V_0 \int d^3x |\psi(\vec{y})|^2 \psi(\vec{x})$$

And so I have that

$$\int d^3x \left[-\frac{\hbar^2 \vec{\nabla}^2}{2m} + U(\vec{x}) - \mu \right] \psi(\vec{x}) + V_0 \int d^3x |\psi(\vec{x})|^2 \psi(\vec{x}) = 0$$

But the Hamiltonian is valid over any arbitrary sub-volume and so locally

$$\left[-\frac{\hbar^2 \vec{\nabla}^2}{2m} + U(\vec{x}) - \mu \right] \psi(\vec{x}) + V_0 |\psi(\vec{x})|^2 \psi(\vec{x}) = 0$$

b) Define new operators $\hat{\psi}(\vec{x}) \to \psi(\vec{x}) + \hat{\eta}(\vec{x})$ where $\psi(\vec{x})$ is the solution to the Gross-Pitaevskii equation and write $\hat{K}$ up to quadratic order in $\hat{\eta}(\vec{x}), \hat{\eta}^+(\vec{x})$. Show that terms linear in $\hat{\eta}(\vec{x}), \hat{\eta}^+(\vec{x})$ are cancelled by $\psi(\vec{x})$ being a solution to the G-P equation.

Again, take:

$$\hat{K} = \hat{H} - \mu \hat{N}$$

$$\hat{K} = \int d^3x \hat{\psi}^+(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu\right]\hat{\psi}(\vec{x}) + \frac{1}{2}\int d^3x d^3y \hat{\psi}^+(\vec{x})\hat{\psi}^+(\vec{y})V(\vec{x}-\vec{y})\hat{\psi}(\vec{y})\hat{\psi}(\vec{x})$$

$$\rightarrow \int d^3x\left(\psi^*(\vec{x}) + \hat{\eta}^+(\vec{x})\right)\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu\right]\left(\psi(\vec{x}) + \hat{\eta}(\vec{x})\right)$$

$$+\frac{1}{2}\int d^3x d^3y\left(\psi^*(\vec{x}) + \hat{\eta}^+(\vec{x})\right)\left(\psi^*(\vec{y}) + \hat{\eta}^+(\vec{y})\right)V_0\delta^3(\vec{x}-\vec{y})\left(\psi(\vec{y}) + \hat{\eta}(\vec{y})\right)\left(\psi(\vec{x}) + \hat{\eta}(\vec{x})\right)$$

$(Linear \quad Only)$

$$\int d^3x \hat{\eta}^+(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu\right]\psi(\vec{x}) + \int d^3x \psi^*(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu\right]\hat{\eta}(\vec{x})$$

$$+\frac{1}{2}\int d^3x d^3y \psi^*(\vec{x})\hat{\eta}^+(\vec{y})V_0\delta^3(\vec{x}-\vec{y})\psi(\vec{y})\psi(\vec{x}) + \frac{1}{2}\int d^3x d^3y \psi^*(\vec{y})\hat{\eta}^+(\vec{x})V_0\delta^3(\vec{x}-\vec{y})\psi(\vec{y})\psi(\vec{x})$$

$$+\frac{1}{2}\int d^3x d^3y \psi^*(\vec{x})\psi^*(\vec{y})V_0\delta^3(\vec{x}-\vec{y})\hat{\eta}(\vec{y})\psi(\vec{x}) + \frac{1}{2}\int d^3x d^3y \psi^*(\vec{x})\psi^*(\vec{y})V_0\delta^3(\vec{x}-\vec{y})\psi(\vec{y})\hat{\eta}(\vec{x})$$

$$= \int d^3x \hat{\eta}^+(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu\right]\psi(\vec{x}) + \int d^3x \psi^*(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu\right]\hat{\eta}(\vec{x})$$

$$+V_0\int d^3x|\psi(\vec{x})|^2\psi(\vec{x})\hat{\eta}^+(\vec{x}) + V_0\int d^3x|\psi(\vec{x})|^2\psi^*(\vec{x})\hat{\eta}(\vec{x})$$

$$= \int d^3x\left[\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu\right]\psi(\vec{x}) + V_0|\psi(\vec{x})|^2\psi(\vec{x})\right]\hat{\eta}^+(\vec{x})$$

$$+\int d^3x \psi^*(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu + V_0|\psi(\vec{x})|^2\right]\hat{\eta}(\vec{x})$$

Note that in the final line of the result, indeed the G-P equation must eliminate the $\hat{\eta}^+(\vec{x})$ term. However, in the second term no such cancellation occurs unless the operator $\hat{\eta}(\vec{x})$ itself obeys the G-P equation as well!

c) Introduce the Bogoliubov transformation:

$$\hat{\varphi}(\vec{x}) = u(\vec{x})\hat{\eta}(\vec{x}) + v(\vec{x})\hat{\eta}^+(\vec{x})$$

$$\hat{\varphi}^+(\vec{x}) = u^*(\vec{x})\hat{\eta}^+(\vec{x}) + v^*(\vec{x})\hat{\eta}(\vec{x})$$

Show that $\left[\hat{\varphi}(\vec{x}), \hat{\varphi}^+(\vec{y})\right] = \delta^3(\vec{x}-\vec{y})$ **if** $|u(\vec{x})|^2 - |v(\vec{x})|^2 = 1$**.**

$$\left[\hat{\varphi}(\vec{x}), \hat{\varphi}^+(\vec{y})\right] = \left[u(\vec{x})\hat{\eta}(\vec{x}) + v(\vec{x})\hat{\eta}^+(\vec{x}), u^*(\vec{y})\hat{\eta}^+(\vec{y}) + v^*(\vec{y})\hat{\eta}(\vec{y})\right]$$

$$= u(\vec{x})u^*(\vec{y})\left[\hat{\eta}(\vec{x}), \hat{\eta}^+(\vec{y})\right] + v(\vec{x})u^*(\vec{y})\left[\hat{\eta}^+(\vec{x}), \hat{\eta}^+(\vec{y})\right] + u(\vec{x})v^*(\vec{y})\left[\hat{\eta}(\vec{x}), \hat{\eta}(\vec{y})\right] + v(\vec{x})v^*(\vec{y})\left[\hat{\eta}^+(\vec{x}), \hat{\eta}(\vec{y})\right]$$

$$Take: \left[\hat{\eta}^+(\vec{x}), \hat{\eta}^+(\vec{y})\right] = \left[\hat{\eta}^+(\vec{x}), \hat{\eta}^+(\vec{y})\right] = 0$$

$$= u(\vec{x})u^*(\vec{y})\left[\hat{\eta}(\vec{x}), \hat{\eta}^+(\vec{y})\right] - v(\vec{x})v^*(\vec{y})\left[\hat{\eta}(\vec{y}), \hat{\eta}^+(\vec{x})\right]$$

Now recognizing that the variables in the second term are dummy variables in any integration over the same space,

$$\left[\hat{\varphi}(\vec{x}),\hat{\varphi}^+(\vec{y})\right]=\left(u(\vec{x})u^*(\vec{y})-v(\vec{y})v^*(\vec{x})\right)\left[\hat{\eta}(\vec{x}),\hat{\eta}^+(\vec{y})\right]$$
$$Take:\left[\hat{\eta}(\vec{x}),\hat{\eta}^+(\vec{y})\right]=\delta^3(\vec{x}-\vec{y})$$
$$\left[\hat{\varphi}(\vec{x}),\hat{\varphi}^+(\vec{y})\right]=\left(u(\vec{x})u^*(\vec{y})-v(\vec{y})v^*(\vec{x})\right)\delta^3(\vec{x}-\vec{y})$$

And so I see based on this expression that $\left[\hat{\varphi}(\vec{x}),\hat{\varphi}^+(\vec{y})\right]=\delta^3(\vec{x}-\vec{y})$ if $|u(\vec{x})|^2-|v(\vec{x})|^2=1$.

d) Write $\hat{K}$ up to quadratic order in $\hat{\eta},\hat{\eta}^+$ found in (b) in terms of $\hat{\varphi},\hat{\varphi}^+$. What is the equation that U,V must obey so that the terms of the form $\hat{\varphi}^2,\hat{\varphi}^{+2}$ are cancelled? These are the Bogoliubov-DeGennes equations!

$$\hat{K}\to\int d^3x\left(\psi^*(\vec{x})+\hat{\eta}^+(\vec{x})\right)\left[-\frac{\hbar^2\vec{\nabla}^2}{2m}+U(\vec{x})-\mu\right]\left(\psi(\vec{x})+\hat{\eta}(\vec{x})\right)$$
$$+\frac{1}{2}\int d^3xd^3y\left(\psi^*(\vec{x})+\hat{\eta}^+(\vec{x})\right)\left(\psi^*(\vec{y})+\hat{\eta}^+(\vec{y})\right)V_0\delta^3(\vec{x}-\vec{y})\left(\psi(\vec{y})+\hat{\eta}(\vec{y})\right)\left(\psi(\vec{x})+\hat{\eta}(\vec{x})\right)$$
$$=\int d^3x\psi^*(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m}+U(\vec{x})-\mu\right]\psi(\vec{x})+\int d^3x\hat{\eta}^+(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m}+U(\vec{x})-\mu\right]\hat{\eta}(\vec{x})$$
$$+\frac{1}{2}\int d^3xd^3y\psi^*(\vec{x})\psi^*(\vec{y})V_0\delta^3(\vec{x}-\vec{y})\psi(\vec{y})\psi(\vec{x})+\frac{1}{2}\int d^3xd^3y\hat{\eta}^+(\vec{y})\hat{\eta}^+(\vec{x})V_0\delta^3(\vec{x}-\vec{y})\psi(\vec{y})\psi(\vec{x})$$
$$+\int d^3xd^3y\hat{\eta}^+(\vec{y})\psi^*(\vec{x})V_0\delta^3(\vec{x}-\vec{y})\psi(\vec{y})\hat{\eta}(\vec{x})+\int d^3xd^3y\hat{\eta}^+(\vec{y})\psi^*(\vec{x})V_0\delta^3(\vec{x}-\vec{y})\hat{\eta}(\vec{y})\psi(\vec{x})$$
$$+\frac{1}{2}\int d^3xd^3y\psi^*(\vec{x})\psi^*(\vec{y})V_0\delta^3(\vec{x}-\vec{y})\hat{\eta}(\vec{y})\hat{\eta}(\vec{x})+h.o.t.$$
$$\approx\int d^3x\psi^*(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m}+U(\vec{x})-\mu\right]\psi(\vec{x})+\int d^3x\hat{\eta}^+(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m}+U(\vec{x})-\mu\right]\hat{\eta}(\vec{x})$$
$$+\frac{V_0}{2}\int d^3x|\psi(\vec{x})|^4+\frac{V_0}{2}\int d^3x\hat{\eta}^{+2}(\vec{x})\psi^2(\vec{x})+2V_0\int d^3x\hat{\eta}^+(\vec{x})\hat{\eta}(\vec{x})|\psi(\vec{x})|^2+\frac{V_0}{2}\int d^3x\psi^{*2}(\vec{x})\hat{\eta}^2(\vec{x})+h.o.t.$$

Now note that, inverting the prescribed transformation, I get:

$$\hat{\eta}(\vec{x})=\hat{\varphi}(\vec{x})u^*(\vec{x})-\hat{\varphi}^+(\vec{x})v(\vec{x})$$
$$\hat{\eta}^+(\vec{x})=\hat{\varphi}^+(\vec{x})u(\vec{x})-\hat{\varphi}(\vec{x})v^*(\vec{x})$$

This inversion can be quickly verified by substitution into the forms for $\hat{\varphi},\hat{\varphi}^+$ given in the problem statement.

Now considering only terms containing the powers $\hat{\eta},\hat{\eta}^+$ from $\hat{K}$, I have:

$$\hat{K}_{relevant} = \int d^3x \hat{\eta}^+(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu\right]\hat{\eta}(\vec{x})$$

$$\frac{V_0}{2}\int d^3x \hat{\eta}^{+2}(\vec{x})\psi^2(\vec{x}) + 2V_0\int d^3x \hat{\eta}^+(\vec{x})\hat{\eta}(\vec{x})|\psi(\vec{x})|^2 + \frac{V_0}{2}\int d^3x \psi^{*2}(\vec{x})\hat{\eta}^2(\vec{x}) + h.o.t.$$

$$\hat{K}_{relevant} \to \int d^3x\left(\hat{\varphi}^+(\vec{x})u(\vec{x}) - \hat{\varphi}(\vec{x})v^*(\vec{x})\right)\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu\right]\left(\hat{\varphi}(\vec{x})u^*(\vec{x}) - \hat{\varphi}^+(\vec{x})v(\vec{x})\right)$$

$$+\frac{V_0}{2}\int d^3x\left(\hat{\varphi}^+(\vec{x})u(\vec{x}) - \hat{\varphi}(\vec{x})v^*(\vec{x})\right)^2\psi^2(\vec{x})$$

$$+2V_0\int d^3x\left(\hat{\varphi}^+(\vec{x})u(\vec{x}) - \hat{\varphi}(\vec{x})v^*(\vec{x})\right)\left(\hat{\varphi}(\vec{x})u^*(\vec{x}) - \hat{\varphi}^+(\vec{x})v(\vec{x})\right)|\psi(\vec{x})|^2$$

$$+\frac{V_0}{2}\int d^3x \psi^{*2}(\vec{x})\left(\hat{\varphi}(\vec{x})u^*(\vec{x}) - \hat{\varphi}^+(\vec{x})v(\vec{x})\right)^2 + h.o.t$$

Now cosidering the $\hat{\varphi}^{+2}$ portion,

$$\hat{K}_{relevant} \to -\int d^3x \hat{\varphi}^+(\vec{x})u(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu\right]\hat{\varphi}^+(\vec{x})v(\vec{x})$$

$$+\frac{V_0}{2}\int d^3x \hat{\varphi}^+(\vec{x})^2 u(\vec{x})^2\psi^2(\vec{x})$$

$$-2V_0\int d^3x \hat{\varphi}^+(\vec{x})^2 u(\vec{x})v(\vec{x})|\psi(\vec{x})|^2$$

$$+\frac{V_0}{2}\int d^3x \psi^{*2}(\vec{x})v(\vec{x})^2\hat{\varphi}^+(\vec{x})^2 + h.o.t$$

By the same argument as in part (a), then, the Hamiltonian is valid across all space and so the the integral must vanish at all sub-regions, so that in order to make this vanish I need:

$$-\hat{\varphi}^+(\vec{x})u(\vec{x})\left[-\frac{\hbar^2\vec{\nabla}^2}{2m} + U(\vec{x}) - \mu\right]\hat{\varphi}^+(\vec{x})v(x) + \frac{V_0}{2}\hat{\varphi}^+(\vec{x})^2 u(\vec{x})^2\psi^2(\vec{x})$$

$$-2V_0\hat{\varphi}^+(\vec{x})^2 u(\vec{x})v(\vec{x})|\psi(\vec{x})|^2 + \frac{V_0}{2}\psi^{*2}(\vec{x})v(\vec{x})^2\hat{\varphi}^+(\vec{x})^2 = 0$$

Where again $\psi(\vec{x})$ is the solution to the original G-P equation, and solutions $\hat{\varphi}^+(\vec{x})$ are found from the unperturbed Hamiltonian. A second constraint is then found using the $\hat{\varphi}^2$ portion of $\hat{K}_{relevant}$, which leads to a coupled pair of differential equations in u, v. Notice that the operator portions can be dropped from this equation after the differential operator acts.