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Quantum Mechanics 3 Homework 10

1) **Consider two coherent eigenstates of the annihilation operator $\hat{a}_0$ with $\hat{a}_0|\alpha_1\rangle = \alpha_1|\alpha_1\rangle$ and $\hat{a}_0|\alpha_2\rangle = \alpha_2|\alpha_2\rangle$ with $\alpha_1 = \sqrt{N_0}e^{i\phi_1}$ and $\alpha_2 = \sqrt{N_0}e^{i\phi_2}$. Show that for $N_0 >>> 1$ the overlap $|\langle\alpha_1|\alpha_2\rangle|^2 \to 0$, namely that these two coherent states are orthogonal in the limit $N_0 >>> 1$ although $\langle\alpha_1|\hat{a}_0^+\hat{a}_0|\alpha_1\rangle = \langle\alpha_2|\hat{a}_0^+\hat{a}_0|\alpha_2\rangle = N_0$, for $\phi_1 \neq \phi_2$.**

Clearly, $\langle\alpha_1|\hat{a}_0^+\hat{a}_0|\alpha_1\rangle = \langle\alpha_2|\hat{a}_0^+\hat{a}_0|\alpha_2\rangle = |\alpha|^2 = N_0$.

For the purposes of this problem, take states normalized so that $\hat{a}^+|n\rangle = |n+1\rangle$, etc.

Now:

$$|\alpha_1\rangle = e^{(\alpha_1\hat{a}_0 - \alpha_1^*\hat{a}_0^+)}|0,0,0,...\rangle$$
$$= e^{-\frac{|\alpha_1|^2}{2}} e^{-\alpha_1\hat{a}_0^+}|0,0,0,...\rangle$$
$$= e^{-\frac{|\alpha_1|^2}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n \alpha_1^n}{\sqrt{n!}} a_1^{+n}|0,0,0,...\rangle$$
$$= e^{-\frac{|\alpha_1|^2}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n!}} \alpha_1^n|n,0,0,...\rangle$$

An analogous result can be obtained for $|\alpha_2\rangle$. Now by substitution:

$$\langle\alpha_1|\alpha_2\rangle = e^{-\frac{|\alpha_1|^2}{2}} e^{-\frac{|\alpha_2|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha_1^n)^* \alpha_2^n}{n!}$$
$$= e^{-\frac{|\alpha_1|^2}{2}} e^{-\frac{|\alpha_2|^2}{2}} \sum_{n=0}^{\infty} \frac{N_0^n}{n!} e^{in(\phi_2-\phi_1)} = e^{-\frac{|\alpha_1|^2}{2}} e^{-\frac{|\alpha_2|^2}{2}} e^{N_0 e^{i(\phi_2-\phi_1)}}$$
$$= e^{N_0\left(e^{i(\phi_2-\phi_1)}-1\right)}$$

Clearly, then, for in-phase $\phi_1 = \phi_2$, the sum will be maximized. For out-of-phase $\phi_1 \neq \phi_2$, then, and large N_0, this will be $(\phi_2 - \phi_1) = 0 : \langle\alpha_1|\alpha_2\rangle = e^{N_0\left(e^{i(\phi_2-\phi_1)}-1\right)} = 1$ and otherwise this will go to zero exponentially.

2) **<u>Bogoliubov Transformations</u>: Consider a Hamiltonian for Bosonic operators $\hat{b}^+, \hat{b}$ of the form $\hat{H} = E(k)\hat{b}_k^+\hat{b}_k + A(k)\left[\hat{b}_k^+\hat{b}_{-k}^+ + \hat{b}_k\hat{b}_{-k}\right]$. Define a Bogoliubov transformation to new Bosonic operators $\hat{\alpha}_k^+, \hat{\alpha}_k$ as follows:**

$$\hat{b}_k = \cosh 2\theta_k \hat{\alpha}_k + \sinh 2\theta_k \hat{\alpha}^+_{-k}$$
$$\hat{b}^+_k = \cosh 2\theta_k \hat{\alpha}^+_k + \sinh 2\theta_k \hat{\alpha}_{-k}$$

a) **Assume that $E(k), A(k), \theta_k$ are all even functions of k and find the form of $\sinh 2\theta_k$ as a function of $A(k), E(k)$ so that $\hat{H} = \Omega(k)\hat{\alpha}^+_k\hat{\alpha}_k + F(k)$ and find $\Omega(k), F(k)$.**

$$\hat{H} = E(k)\hat{b}^+_k\hat{b}_k + A(k)\left[\hat{b}^+_k\hat{b}^+_{-k} + \hat{b}_k\hat{b}_{-k}\right]$$
$$Note: \sinh(-x) = -\sinh(x) \quad \cosh(-x) = \cosh(x)$$
$$= E(k)\left[\cosh 2\theta_k \hat{\alpha}^+_k + \sinh 2\theta_k \hat{\alpha}_{-k}\right]\left[\cosh 2\theta_k \hat{\alpha}_k + \sinh 2\theta_k \hat{\alpha}^+_{-k}\right]$$
$$+ A(k)\begin{bmatrix} \left(\cosh 2\theta_k \hat{\alpha}^+_k + \sinh 2\theta_k \hat{\alpha}_{-k}\right)\left(\cosh 2\theta_k \hat{\alpha}^+_{-k} + \sinh 2\theta_k \hat{\alpha}_k\right) \\ + \left(\cosh 2\theta_k \hat{\alpha}_k + \sinh 2\theta_k \hat{\alpha}^+_{-k}\right)\left(\cosh 2\theta_k \hat{\alpha}_{-k} + \sinh 2\theta_k \hat{\alpha}^+_k\right) \end{bmatrix}$$
$$= E(k)\left[\cosh^2 2\theta_k \hat{\alpha}^+_k\hat{\alpha}_k + \cosh 2\theta_k \sinh 2\theta_k \hat{\alpha}^+_k\hat{\alpha}^+_{-k} + \cosh 2\theta_k \sinh 2\theta_k \hat{\alpha}_{-k}\hat{\alpha}_k + \sinh^2 2\theta_k \hat{\alpha}_{-k}\hat{\alpha}^+_{-k}\right]$$
$$+ A(k)\begin{bmatrix} \cosh^2 2\theta_k \hat{\alpha}^+_k\hat{\alpha}^+_{-k} + \sinh^2 2\theta_k \hat{\alpha}_{-k}\hat{\alpha}_k + \cosh 2\theta_k \sinh 2\theta_k \hat{\alpha}^+_k\hat{\alpha}_k + \cosh 2\theta_k \sinh 2\theta_k \hat{\alpha}_{-k}\hat{\alpha}^+_{-k} \\ \cosh^2 2\theta_k \hat{\alpha}_k\hat{\alpha}_{-k} + \sinh^2 2\theta_k \hat{\alpha}^+_{-k}\hat{\alpha}^+_k + \cosh 2\theta_k \sinh 2\theta_k \hat{\alpha}_k\hat{\alpha}^+_k + \cosh 2\theta_k \sinh 2\theta_k \hat{\alpha}^+_{-k}\hat{\alpha}_{-k} \end{bmatrix}$$

Take: $\left[\hat{\alpha}_k, \hat{\alpha}^+_k\right] = 1$. This will be shown in part (b). Then,

$$= E(k)\left[\cosh^2 2\theta_k \hat{\alpha}^+_k\hat{\alpha}_k + \cosh 2\theta_k \sinh 2\theta_k \hat{\alpha}^+_k\hat{\alpha}^+_{-k} + \cosh 2\theta_k \sinh 2\theta_k \hat{\alpha}_k\hat{\alpha}_{-k} + \sinh^2 2\theta_k \hat{\alpha}^+_{-k}\hat{\alpha}_{-k} + \sinh^2 2\theta_k\right]$$
$$+ A(k)\begin{bmatrix} \cosh^2 2\theta_k \hat{\alpha}^+_k\hat{\alpha}^+_{-k} + \sinh^2 2\theta_k \hat{\alpha}_k\hat{\alpha}_{-k} + \cosh 2\theta_k \sinh 2\theta_k \hat{\alpha}^+_k\hat{\alpha}_k + \cosh 2\theta_k \sinh 2\theta_k \hat{\alpha}^+_{-k}\hat{\alpha}_{-k} + \cosh 2\theta_k \sinh 2\theta_k \\ \cosh^2 2\theta_k \hat{\alpha}_k\hat{\alpha}_{-k} + \sinh^2 2\theta_k \hat{\alpha}^+_k\hat{\alpha}^+_{-k} + \cosh 2\theta_k \sinh 2\theta_k \hat{\alpha}^+_k\hat{\alpha}_k + \cosh 2\theta_k \sinh 2\theta_k \hat{\alpha}^+_{-k}\hat{\alpha}_{-k} + \cosh 2\theta_k \sinh 2\theta_k \end{bmatrix}$$
$$= \left[E(k)\cosh^2 2\theta_k + 2A(k)\cosh 2\theta_k \sinh 2\theta_k\right]\hat{\alpha}^+_k\hat{\alpha}_k + \left[E(k)\cosh 2\theta_k \sinh 2\theta_k + A(k)\cosh^2 2\theta_k + A(k)\sinh^2 2\theta_k\right]\hat{\alpha}^+_k\hat{\alpha}^+_{-k}$$
$$+ \left[E(k)\cosh 2\theta_k \sinh 2\theta_k + A(k)\cosh^2 2\theta_k + A(k)\sinh^2 2\theta_k\right]\hat{\alpha}_k\hat{\alpha}_{-k}$$
$$+ \left[E(k)\sinh^2 2\theta_k + 2A(k)\cosh 2\theta_k \sinh 2\theta_k\right]\hat{\alpha}^+_{-k}\hat{\alpha}_{-k} + \sinh^2 2\theta_k$$
$$= \left[E(k)\cosh^2 2\theta_k + A(k)\sinh 4\theta_k\right]\hat{\alpha}^+_k\hat{\alpha}_k + \left[E(k)\frac{\sinh 4\theta_k}{2} + A(k)\cosh 4\theta_k\right]\hat{\alpha}^+_k\hat{\alpha}^+_{-k}$$
$$+ \left[E(k)\frac{\sinh 4\theta_k}{2} + A(k)\cosh 4\theta_k\right]\hat{\alpha}_k\hat{\alpha}_{-k}$$
$$+ \left[E(k)\sinh^2 2\theta_k + A(k)\sinh 4\theta_k\right]\hat{\alpha}^+_{-k}\hat{\alpha}_{-k} + \sinh^2 2\theta_k E(k)$$

So I see that in order to fix θ_k, I need:

$$\frac{A(k)}{E(k)} = -\frac{\sinh 4\theta_k}{2\cosh 4\theta_k}.$$

And the remaining terms will be:

$$\Omega(+k) = E(k)\cosh^2 2\theta_k + A(k)\sinh 4\theta_k$$
$$\Omega(-k) = E(k)\sinh^2 2\theta_k + A(k)\sinh 4\theta_k$$
$$F(k) = \sinh^2 2\theta_k E(k)$$

b) Show that if $\left[\hat{b}_k^+, \hat{b}_k\right] = 1; \left[\hat{b}_k^+, \hat{b}_k^+\right] = \left[\hat{b}_k, \hat{b}_k\right] = 0$ **, then** $\left[\hat{\alpha}_k^+, \hat{\alpha}_k\right] = 1; \left[\hat{\alpha}_k^+, \hat{\alpha}_k^+\right] = \left[\hat{\alpha}_k, \hat{\alpha}_k\right] = 0$ **.**

$$\left[\hat{b}_k^+, \hat{b}_k\right] = 1 = \left[\cosh 2\theta_k \hat{\alpha}_k^+ + \sinh 2\theta_k \hat{\alpha}_{-k}, \cosh 2\theta_k \hat{\alpha}_k + \sinh 2\theta_k \hat{\alpha}_{-k}^+\right]$$
$$= \cosh^2 2\theta_k \left[\hat{\alpha}_k^+, \hat{\alpha}_k\right] + \sinh^2 2\theta_k \left[\hat{\alpha}_{-k}, \hat{\alpha}_{-k}^+\right]$$
$$\textit{Substitute} \quad \left[\hat{\alpha}_k^+, \hat{\alpha}_k\right] = 1:$$
$$= \cosh^2 2\theta_k - \sinh^2 2\theta_k = 1$$

So I have verified that $\left[\hat{\alpha}_k^+, \hat{\alpha}_k\right] = 1$ satisfies $\left[\hat{b}_k^+, \hat{b}_k\right] = 1$.

All operators necessarily commute with themselves, so the identities $\left[\hat{\alpha}_k^+, \hat{\alpha}_k^+\right] = \left[\hat{\alpha}_k, \hat{\alpha}_k\right] = 0$ are necessarily true.

c) Show that if the operators $\hat{b}^+, \hat{b}$ **are Fermionic instead of bosonic, the Bogoliubov transformation must be** $\cosh\theta_k \to \cos\theta_k \quad \sinh\theta_k \to \sin\theta_k$ **so the new operators** $\hat{\alpha}_k^+, \hat{\alpha}_k$ **obey anticommutation relations.**

In the case of Fermionic operators, then:

$$\hat{H} = E(k)\hat{b}_k^+\hat{b}_k + A(k)\left[\hat{b}_k^+\hat{b}_{-k}^+ + \hat{b}_k\hat{b}_{-k}\right]$$
$$\textit{Note}: \sin(-x) = -\sin(x) \quad \cos(-x) = \cos(x)$$
$$= E(k)\left[\cos 2\theta_k \hat{\alpha}_k^+ + \sin 2\theta_k \hat{\alpha}_{-k}\right]\left[\cos 2\theta_k \hat{\alpha}_k + \sin 2\theta_k \hat{\alpha}_{-k}^+\right]$$
$$+ A(k)\begin{bmatrix} \left(\cos 2\theta_k \hat{\alpha}_k^+ + \sin 2\theta_k \hat{\alpha}_{-k}\right)\left(\cos 2\theta_k \hat{\alpha}_{-k}^+ + \sin 2\theta_k \hat{\alpha}_k\right) \\ + \left(\cos 2\theta_k \hat{\alpha}_k + \sin 2\theta_k \hat{\alpha}_{-k}^+\right)\left(\cos 2\theta_k \hat{\alpha}_{-k} + \sin 2\theta_k \hat{\alpha}_k^+\right) \end{bmatrix}$$
$$= E(k)\left[\cos^2 2\theta_k \hat{\alpha}_k^+\hat{\alpha}_k + \cos 2\theta_k \sin 2\theta_k \hat{\alpha}_k^+\hat{\alpha}_{-k}^+ + \cos 2\theta_k \sin 2\theta_k \hat{\alpha}_{-k}\hat{\alpha}_k + \sin^2 2\theta_k \hat{\alpha}_{-k}\hat{\alpha}_{-k}^+\right]$$
$$+ A(k)\begin{bmatrix} \cos^2 2\theta_k \hat{\alpha}_k^+\hat{\alpha}_{-k}^+ + \sin^2 2\theta_k \hat{\alpha}_{-k}\hat{\alpha}_k + \cos 2\theta_k \sin 2\theta_k \hat{\alpha}_k^+\hat{\alpha}_k + \cos 2\theta_k \sin 2\theta_k \hat{\alpha}_{-k}\hat{\alpha}_{-k}^+ \\ \cos^2 2\theta_k \hat{\alpha}_k\hat{\alpha}_{-k} + \sin^2 2\theta_k \hat{\alpha}_{-k}^+\hat{\alpha}_k^+ + \cos 2\theta_k \sin 2\theta_k \hat{\alpha}_k\hat{\alpha}_k^+ + \cos 2\theta_k \sin 2\theta_k \hat{\alpha}_{-k}^+\hat{\alpha}_{-k} \end{bmatrix}$$

Take: $\left\{\hat{\alpha}_k, \hat{\alpha}_k^+\right\} = 1$. Then,

$$
\begin{aligned}
&= E(k)\left[\cos^2 2\theta_k \hat{\alpha}_k^+ \hat{\alpha}_k + \cos 2\theta_k \sin 2\theta_k \hat{\alpha}_k^+ \hat{\alpha}_{-k}^+ - \cos 2\theta_k \sin 2\theta_k \hat{\alpha}_k \hat{\alpha}_{-k} - \sin^2 2\theta_k \hat{\alpha}_{-k}^+ \hat{\alpha}_{-k} + \sin^2 2\theta_k\right] \\
&+ A(k)\begin{bmatrix} \cos^2 2\theta_k \hat{\alpha}_k^+ \hat{\alpha}_{-k}^+ - \sin^2 2\theta_k \hat{\alpha}_k \hat{\alpha}_{-k} + \cos 2\theta_k \sin 2\theta_k \hat{\alpha}_k^+ \hat{\alpha}_k - \cos 2\theta_k \sin 2\theta_k \hat{\alpha}_{-k}^+ \hat{\alpha}_{-k} + \cos 2\theta_k \sin 2\theta_k \\ \cos^2 2\theta_k \hat{\alpha}_k \hat{\alpha}_{-k} - \sin^2 2\theta_k \hat{\alpha}_k^+ \hat{\alpha}_{-k}^+ - \cos 2\theta_k \sin 2\theta_k \hat{\alpha}_k^+ \hat{\alpha}_k + \cos 2\theta_k \sin 2\theta_k \hat{\alpha}_{-k}^+ \hat{\alpha}_{-k} - \cos 2\theta_k \sin 2\theta_k \end{bmatrix} \\
&= \left[E(k)\cos^2 2\theta_k\right]\hat{\alpha}_k^+ \hat{\alpha}_k + \left[E(k)\cos 2\theta_k \sin 2\theta_k + A(k)\cos^2 2\theta_k - A(k)\sin^2 2\theta_k\right]\hat{\alpha}_k^+ \hat{\alpha}_{-k}^+ \\
&+ \left[-E(k)\cos 2\theta_k \sin 2\theta_k + A(k)\cos^2 2\theta_k - A(k)\sin^2 2\theta_k\right]\hat{\alpha}_k \hat{\alpha}_{-k} \\
&+ \left[-E(k)\sin^2 2\theta_k\right]\hat{\alpha}_{-k}^+ \hat{\alpha}_{-k} + \sin^2 2\theta_k E(k)
\end{aligned}
$$

And now clearly the two-creation and two-annihilation terms can be eliminated by requiring:

$$
-E(k)\cos 2\theta_k \sin 2\theta_k + A(k)\cos^2 2\theta_k - A(k)\sin^2 2\theta_k = 0
$$

3) **<u>The Lamb Shift</u>: Consider the energy shift of hydrogenic atomic levels ε_n in second order in the electromagnetic interactions.**

a) **Find the complete expression for $\Delta\varepsilon_n^{(2)}$ and show that it is divergent.**

The total Hamiltonian is given by the sum of the mechanical, electromagnetic and interaction Hamiltonians:

$$
\begin{aligned}
&\hat{H} = \hat{H}_0^{at} + \hat{H}_0^{em} + \hat{H}_I \\
&\hat{H}_I \approx \hat{H}_I^{(1)} + \hat{H}_I^{(2)} \qquad \hat{\vec{p}} = -i\hbar\vec{\nabla}_x \\
&\hat{H}_I^{(1)} = -\frac{e}{mc}\int d^3x\,\psi^+(\vec{x})\vec{A}(\vec{x})\hat{\vec{p}}\,\psi(\vec{x}) \\
&\hat{H}_I^{(2)} = \frac{e^2}{mc^2}\int d^3x\,\psi^+(\vec{x})\vec{A}^2(\vec{x})\psi(\vec{x})
\end{aligned}
$$

Then considering the perturbation due to the interaction portion, I obtain:

$$
\Delta\varepsilon_n = \langle n;0|\hat{H}_I^{(1)}|n;0\rangle + \langle n;0|\hat{H}_I^{(2)}|n;0\rangle + \sum_{m\neq n}\frac{\left|\langle n;0|\hat{H}_I^{(1)}|m\rangle\right|^2}{\varepsilon_n - \varepsilon_m + i\eta}
$$

Above, $|m\rangle$ is a state with the electron in level l plus a photon.

In homework 6, problem 4, I addressed the first-order shifts which were seen to be the same for every energy . In problem 6b, I obtained:

$$
\langle n;0|\hat{H}_I^{(1)}|n;0\rangle + \langle n;0|\hat{H}_I^{(2)}|n;0\rangle = \frac{e^2\hbar}{4\pi^2 mcV}\int_0^\infty k\,dk
$$

Is there a problem here? For one thing, I would have hoped that volume in the denominator would have been eliminated in a conversion to k-space, but since my integrals were initially over momentum space, the volume arising from the definition of the potential never had an opportunity to disappear. Second, I would have liked to see an h-bar in the denominator. I think I may have lost a factor of $\frac{1}{\hbar^3}$ *while collapsing the delta-functions. Please refer to homework 6 problem 4.*

At any rate, I ignore the infinite but constant shift arising from these two terms. It is left to evaluate the portion

$$\sum_{m\neq n}\frac{\left|\langle n;0|\hat{H}_I^{(1)}|m\rangle\right|^2}{\varepsilon_n-\varepsilon_m+i\eta}.$$

Take:

$$\langle m|\hat{H}_I^{(1)}|n;0\rangle=\left(\frac{-e}{mc}\right)\sum_{l,l'}\sum_{k,\lambda}\sqrt{\frac{\hbar c^2}{2V\omega(k)}}\vec{\varepsilon}_\lambda(k)\cdot\int d^3xU_l^*(\vec{x})\hat{\vec{p}}U_{l'}(\vec{x})e^{i\vec{k}\cdot\vec{x}}\langle m|\hat{c}_l^+\hat{c}_l\hat{a}_{k,\lambda}^+|n;0\rangle.$$

As discussed above, since $\langle m|\rightarrow\langle l;1_{k,\lambda}|$, yielding the identifications used below.

In the dipole approximation, we take $e^{i\vec{k}\cdot\vec{x}}\approx 1$, so that

$$\int d^3xU_l^*(\vec{x})\hat{\vec{p}}U_n(\vec{x})e^{i\vec{k}\cdot\vec{x}}\rightarrow\int d^3xU_l^*(\vec{x})\hat{\vec{p}}U_n(\vec{x})\equiv\vec{p}_{\ln}$$

Now $\varepsilon_m=\varepsilon_l+\hbar\omega(k)$, and

$$\sum_{m\neq n}\frac{\left|\langle n;0|\hat{H}_I^{(1)}|m\rangle\right|^2}{\varepsilon_n-\varepsilon_m+i\eta}=\left(\frac{e}{mc}\right)^2\sum_l\int\frac{d^3k}{(2\pi)^3}\sum_\lambda\frac{1}{2\omega(k)}\frac{\left|\vec{\varepsilon}_\lambda(k)\cdot\vec{p}_{\ln}\right|^2}{\varepsilon_n-\varepsilon_l-\hbar ck+i\eta}$$

Geometrically,

$$\sum_\lambda\left|\vec{\varepsilon}_\lambda(k)\cdot\vec{p}_{\ln}\right|^2=\left|\vec{p}_{\ln}\right|^2\left(1-\cos^2\theta\right)$$

$$\int d^3k=2\pi\int k^2dk\int_{-1}^{1}d(\cos\theta)$$

$$\int_{-1}^{1}d(\cos\theta)\left(1-\cos^2\theta\right)=\frac{4}{3}$$

Now,

$$\sum_{m\neq n}\frac{\left|\langle n;0|\hat{H}_I^{(1)}|m\rangle\right|^2}{\varepsilon_n-\varepsilon_m+i\eta}=-\frac{2}{3\pi}\frac{\alpha}{m^2c^2}\int_0^\infty kdk\sum_l\frac{|\vec{p}_{\ln}|^2}{\hbar ck+\varepsilon_l-\varepsilon_n-i\eta}$$

I have taken $\omega(k)=\hbar ck$, and the fine structure constant $\alpha=\frac{e^2}{4\pi\hbar c}$.

Now the denominator dies like $\frac{1}{\hbar ck}$ and the integral diverges, for

$$\Delta\varepsilon_n^{div}=-\frac{2}{3\pi}\frac{\alpha}{m^2c^2}\sum_l|\vec{p}_{\ln}|^2\int_0^\infty\frac{kdk}{\hbar ck}=\left[-\frac{1}{3\pi}\frac{4\alpha}{c^2}\int_0^\infty\frac{kdk}{\hbar ck}\right]\frac{\langle n|\hat{\vec{p}}^2|n\rangle}{2m^2}$$

b) In the atomic Hamiltonian write the mass of the electron as $m=m_{ph}-\delta m$ and use $\frac{\delta m}{m_{ph}}<<1$ with m_{ph} the physically measured electron mass and δm a power series expansion $\delta m=\delta m^{(2)}\alpha+....$. Treat $\int d^3x\psi^+\frac{\hat{\vec{p}}^2}{m_{ph}^2}\psi\delta m$ as a perturbation and obtain the first order shift in the energy (first order in δm) of hydrogenic levels caused by this perturbation.

The atomic Hamiltonian is

$$\hat{H}_{at}=\frac{\hat{\vec{p}}^2}{2m}+V(\vec{x}),$$

with $V(\vec{x})$ the Coulomb potential of a proton.

Taylor expanding the mass-dependent portion, then, I have:

$$\int d^3x\psi^+ \frac{\hat{\vec{p}}^2}{2m}\psi$$

$$\approx \int d^3x\psi^+ \frac{\hat{\vec{p}}^2}{2m_{ph}}\psi + \int d^3x\psi^+ \frac{\hat{\vec{p}}^2}{2m_{ph}{}^2}\psi\delta m + ...$$

$$\approx \int d^3x\psi^+ \frac{\hat{\vec{p}}^2}{2m_{ph}}\psi + \int d^3x\psi^+ \frac{\hat{\vec{p}}^2}{2m_{ph}{}^2}\psi\delta m + ...$$

$$E \approx \left(1+\frac{\delta m}{m_{ph}}\right)\int d^3x\psi^+ \frac{\hat{\vec{p}}^2}{2m_{ph}}\psi.+.\int d^3x\psi^+ V(\vec{x})\psi.$$

Then, for a Hydrogenic level,

$$\Delta E_n^{(1)} = \frac{\delta m}{2m_{ph}{}^2}\sum_l |p_{\ln}|^2$$

c) Mass Renormalization: Fix δm by requesting that the energy shift found above in (b) cancels the divergent contribution to the energy shift found in (a). Take the resulting integrals between $0 \le k \le \frac{m_{ph}c}{\hbar}$ and obtain the final expression for the energy shift.

For a free electron, then, the energy shift is

$$\Delta E_n^{(1)} \approx \langle n;0|\int d^3x\psi^+ \frac{\hat{\vec{p}}^2}{2m_{ph}{}^2}\psi.|n;0\rangle\delta m$$

From part (a), I had:

$$\Delta\varepsilon_n^{div} = \left[-\frac{1}{3\pi}\frac{4\alpha}{c^2}\int_0^\infty \frac{kdk}{\hbar ck}\right]\frac{\langle n|\hat{\vec{p}}^2|n\rangle}{2m^2}$$

Then, asking the solution from part (b) to cancel this, I have:

$$\Delta\varepsilon_n^{div} = -\langle n;0|\int d^3x\psi^+\left[-\frac{\hat{\vec{p}}^2}{2m_{ph}{}^2}\right]\psi.|n;0\rangle\delta m = \left[-\frac{1}{3\pi}\frac{4\alpha}{c^2}\int_0^\infty \frac{kdk}{\hbar ck}\right]\frac{\langle n|\hat{\vec{p}}^2|n\rangle}{2m^2}$$

$$\delta m = \frac{1}{3\pi}\frac{4\alpha}{c^2}\int_0^\infty \frac{kdk}{\hbar ck} = \frac{4}{3\pi}\frac{\alpha\hbar^2}{c^2}\int_0^\infty \frac{kdk}{\hbar ck}$$

Taking the Hydrogenic shift from part (b), and subtracting off the divergent part then:

$$\Delta\varepsilon_n = \frac{4}{3\pi}\frac{\alpha\hbar^2}{2m_p^2}\int_0^\infty \frac{kdk}{\hbar ck}\sum_l |p_{\ln}|^2 - \frac{4}{3\pi}\frac{\alpha\hbar^2}{2m_p^2}\int_0^\infty \frac{kdk}{\hbar ck}\sum_l \frac{|p_{\ln}|^2}{\hbar ck + \varepsilon_l - \varepsilon_n + i\eta}$$

Combining these and limiting the integral as prescribed in the problem statement, I get:

$$\Delta\varepsilon_n = -\frac{4\alpha\hbar}{3\pi^2 m_p^2 c}\int_0^{\frac{m_p c}{\hbar}} dk \sum_l \frac{|p_{\ln}|(\varepsilon_l - \varepsilon_n)}{[\hbar ck + \varepsilon_l - \varepsilon_n - i\eta]}$$

$$= -\frac{4\alpha\hbar}{3\pi^2 m_p^2 c}\sum_l |p_{\ln}|^2 \frac{\varepsilon_l - \varepsilon_n}{\hbar}\ln\left|\frac{\varepsilon_l - \varepsilon_n}{m_p c^2}\right|$$

4) Plasmons in 2-dimensional electron gases: Graphene is a nearly two-dimensional electron gas, where the Fourier transform of the Coulomb interaction is $V_q = \frac{2\pi e^2}{q}$ (not $\frac{4\pi e^2}{q^2}$).

a) Show that plasmon excitations at long wavelengths feature the dispersion relation $\omega(q) = \sqrt{\frac{2\pi e^2 n}{m} q}$.

Analogously to homework 9 problem 1, I consider a free electron gas in its ground state.

Here, the density operator is $\rho_{\vec{q}} = \sum_{p,\sigma} a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}$.

While we wish to examine density fluctuations and thus the equation of motion of density, I will consider the particle-hole operator, a component $a^+_{\vec{p}+\vec{q},\sigma} a_{\vec{p},\sigma}$ of the density operator.

Then, the single-particle Hamiltonian is

$$H_0 = \sum_p \varepsilon(p) a^+_{\vec{p}} a_{\vec{p}} ,$$

The interaction Hamiltonian is:

$$H_I = \frac{1}{2V}\sum_k V_k \sum_{p,q,\sigma_1,\sigma_2} a^+_{\vec{p}+\vec{k},\sigma_1} a^+_{\vec{q}-\vec{k},\sigma_2} a_{\vec{q},\sigma_2} a_{\vec{p},\sigma_1}$$

Above, V_k is the Coulomb interaction.

Obtaining the equation of motion for this operator, then, I have:

$$-i\hbar\frac{d}{dt}\left(a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}\right)=\left[H_0+H_I,a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}\right]=\left[H_0,a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}\right]+\left[H_I,a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}\right]$$

$$\left[H_0,a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}\right]=\left[\sum_{p'}\varepsilon(p')a^{+}_{\vec{p}'}a_{\vec{p}'},a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}\right]=\sum_{p'}\varepsilon(p')\left[a^{+}_{\vec{p}'}a_{\vec{p}'},a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}\right]$$
$$=\sum_{p'}\varepsilon(p')\delta_{\vec{p}',\vec{p}+\vec{q}}a^{+}_{\vec{p}',\sigma}a_{\vec{p},\sigma}+\sum_{p'}\varepsilon(p')\delta_{\vec{p}',\vec{q}}a_{\vec{p}',\sigma}a^{+}_{\vec{p}+\vec{q},\sigma}=\varepsilon(\vec{p}+\vec{q})a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}-\varepsilon(\vec{p})a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}$$
$$=\left(\varepsilon(\vec{p}+\vec{q})-\varepsilon(\vec{p})\right)a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}$$

$$\left[H_I,a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}\right]=\left[\frac{1}{2V}\sum_k V_k\sum_{p',q',\sigma_1,\sigma_2}a^{+}_{\vec{p}'+\vec{k},\sigma_1}a^{+}_{\vec{q}'-\vec{k},\sigma_2}a_{\vec{q}',\sigma_2}a_{\vec{p}',\sigma_1},a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}\right]$$
$$=\frac{1}{2V}\sum_k V_k\left\{\left(a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p}+\vec{k},\sigma}-a^{+}_{\vec{p}+\vec{q}-\vec{k},\sigma}a_{\vec{p},\sigma}\right)\rho_q+\rho_q\left(a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p}+\vec{k},\sigma}-a^{+}_{\vec{p}+\vec{q}-\vec{k},\sigma}a_{\vec{p},\sigma}\right)\right\}$$

In order to linearize this relationship, I use the Random Phase Approximation: namely, then, with $\left|FS\right\rangle$ the Fermi sphere, that

$$a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p}+\vec{k},\sigma}\rightarrow\left\langle FS\right|a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p}+\vec{k},\sigma}\left|FS\right\rangle=n_{\vec{p}+\vec{k}}\delta_{\vec{k},\vec{q}}$$
$$a^{+}_{\vec{p}+\vec{q}-\vec{k},\sigma}a_{\vec{p},\sigma}\rightarrow\left\langle FS\right|a^{+}_{\vec{p}+\vec{q}-\vec{k},\sigma}a_{\vec{p},\sigma}\left|FS\right\rangle=n_{\vec{p}}\delta_{\vec{k},\vec{q}}$$

Now since any collective excitation in charge density has a time evolution like $e^{i\omega(k)t}e^{-\frac{\gamma(k)}{\hbar}t}$, I can apply the derivative to write:

$$-i\hbar\frac{d}{dt}\left(a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}\right)=\left[H_0+H_I,a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}\right]$$
$$\left(\hbar\omega+i\gamma\right)a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}=\left(\varepsilon(\vec{p}+\vec{q})-\varepsilon(\vec{p})\right)a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}-\frac{V_q}{V}\left(n_{p+q}-n_p\right)\rho_q$$
$$a^{+}_{\vec{p}+\vec{q},\sigma}a_{\vec{p},\sigma}=\frac{V_q}{V}\frac{n_p-n_{p+q}}{\hbar\omega+i\gamma-\varepsilon(\vec{p}+\vec{q})+\varepsilon(\vec{p})}\rho_k$$

Summing over all $\vec{p}$ on both sides,

$$\rho_q = \frac{V_q}{V}\sum_{\vec{p}} \frac{n_p - n_{p+q}}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})}\rho_q$$

$$1 = \frac{V_q}{V}\sum_{\vec{p}} \frac{n_p - n_{p+q}}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})}$$

$$1 = V_q \int \frac{d^2 p}{(2\pi)^2} \frac{n_p - n_{p+q}}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})}$$

$$1 = V_q \int \frac{d^2 p}{(2\pi)^2} \left[\frac{n_p}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})} - \frac{n_{p+q}}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})} \right]$$

Re-indexing,

$$1 = V_q \int \frac{d^2 p}{(2\pi)^2} \left[\frac{n_p}{\hbar\omega + i\gamma - \varepsilon(\vec{p}+\vec{q}) + \varepsilon(\vec{p})} - \frac{n_p}{\hbar\omega + i\gamma - \varepsilon(\vec{p}) + \varepsilon(\vec{p}-\vec{q})} \right]$$

Using

$$\varepsilon(\vec{p}) \to \frac{\hbar^2 p^2}{2m} \quad \varepsilon(\vec{p}+\vec{q}) \to \frac{\hbar^2 p^2}{2m} + \frac{\hbar^2 q^2}{2m} - \frac{pq\cos\theta\hbar^2}{m}$$

where θ is the angle between $\vec{p}$ and $\vec{q}$, I may then write:

$$1 = V_q \int \frac{d^2 p}{(2\pi)^2} n_p \frac{q^2\hbar^2}{m} \frac{1}{\left(\hbar\omega + i\gamma - \frac{pq\cos\theta\hbar^2}{m}\right)^2 - \left(\frac{q^2\hbar^2}{2m}\right)^2}$$

Consider undamped oscillations with $\gamma \to 0$. Further, take the two-dimensional potential $V_q = \frac{2\pi e^2}{q}$. Then I get:

$$1=\frac{2\pi e^2\hbar^2 q}{m}\int\frac{d^2p}{(2\pi)^2}n_p\frac{1}{\left(\hbar\omega-\frac{pq\cos\theta\hbar^2}{m}\right)^2-\left(\frac{q^2\hbar^2}{2m}\right)^2}$$

$$1=\frac{2\pi e^2\hbar^2 q}{m\omega^2}\int\frac{d^2p}{(2\pi)^2}n_p\frac{1}{\left(\hbar-\frac{pq\cos\theta\hbar^2}{m\omega}\right)^2-\left(\frac{q^2\hbar^2}{2m\omega}\right)^2}$$

$$Expand \quad in \quad \frac{q}{\omega}$$

$$1=\frac{2\pi e^2\hbar^2 q}{m\omega^2}\int\frac{d^2p}{(2\pi)^2}n_p\left[\frac{1}{\hbar^2}+...\right]$$

$$1=\frac{2\pi e^2 q}{m\omega^2}n$$

$$\omega=\sqrt{\frac{2\pi e^2 q}{m}}$$

b) The static dielectric function is $\varepsilon(q,0)=1+\frac{\kappa}{q}$; find κ.

Taking the dielectric function from the Random Phase Approximation, I have:

$$\varepsilon(\vec{q},\omega)=1-2V_q\int\frac{d^2p}{(2\pi)^2}\frac{n_p-n_{\vec{p}+\vec{q}}}{\hbar\omega-\varepsilon(\vec{p}+\vec{q})+\varepsilon(\vec{p})}$$

$$\varepsilon(\vec{q},0)=1-\frac{4\pi e^2}{q}\int\frac{d^2p}{(2\pi)^2}\frac{n_p-n_{\vec{p}+\vec{q}}}{\varepsilon(\vec{p})-\varepsilon(\vec{p}+\vec{q})}$$

$$=1+\frac{4\pi e^2}{q}\int\frac{d^2p}{(2\pi)^2}\frac{n_p}{\varepsilon(\vec{p}+\vec{q})-\varepsilon(\vec{p})}=1+\frac{e^2}{q\pi}\int pdpd\theta\frac{n_p}{\frac{\hbar^2q^2}{2m}-\frac{\hbar^2qp\cos\theta}{m}}$$

$$=1+\frac{2e^2}{q}\int pdp\frac{n_p}{\sqrt{\left(\frac{\hbar^2q^2}{2m}\right)^2-\left(\frac{\hbar^2qp}{m}\right)^2}}=1+\frac{2me^2}{q^2\hbar^2}\int pdp\frac{n_p}{\sqrt{\left(\frac{q}{2}\right)^2-(p)^2}}$$

$$=1+\frac{2me^2}{q^2\hbar^2}\left[\frac{q-\sqrt{q^2-4k_F^2}}{2}\right]=1+\frac{me^2}{q\hbar^2}\left[1-\sqrt{1-\frac{4k_F^2}{q^2}}\right]$$