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Quantum Mechanics 3 Homework 1

1) **Consider two quantum mechanical operators** $\hat{A}, \hat{B}$ **.**
a) **Show the formula**

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \left[\hat{A},\hat{B}\right] + \frac{1}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \frac{1}{3!}\left[\hat{A},\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]\right] + \ldots + \frac{1}{n!}\left[\hat{A},.(n \quad commutators).,\left[\hat{A},\hat{B}\right]\right]$$

Now consider the Taylor expansion

$$f(\alpha) = e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = \hat{B} + \alpha\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right) + \frac{1}{2!}\alpha^2\left(\hat{A}\hat{A}\hat{B} - 2\hat{A}\hat{B}\hat{A} + \hat{B}\hat{A}\hat{A}\right)$$

$$+\frac{1}{3!}\alpha^3\left(\hat{A}\hat{A}\hat{A}\hat{B} - 3\hat{A}\hat{A}\hat{B}\hat{A} + 3\hat{A}\hat{A}\hat{B}\hat{A} - \hat{B}\hat{A}\hat{A}\hat{A}\right) + \ldots + \frac{1}{n!}\alpha^n\sum_{i=0}^{n}\binom{n}{i}(-1)^i\hat{A}^{n-i}\hat{B}\hat{A}^i$$

or, letting $\alpha = 1$,

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\hat{A}^{n-i}\hat{B}\hat{A}^i$$

Now all that's left is to show that the n-commutator

$$\left[\hat{A},.(n \quad commutators).,\left[\hat{A},\hat{B}\right]\right] = \sum_{i=0}^{n}\binom{n}{i}(-1)^i\hat{A}^{n-i}\hat{B}\hat{A}^i \,.$$

Now consider a proof by recursion. It is clear from the Taylor expansion above that these basis cases work:

$$n = 0 : \hat{B} = \hat{B} = \sum_{i=0}^{0}\binom{0}{i}(-1)^i\hat{A}^{0-i}\hat{B}\hat{A}^i$$

$$n = 1 : \left[\hat{A},\hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A} = \sum_{i=0}^{1}\binom{1}{i}(-1)^i\hat{A}^{1-i}\hat{B}\hat{A}^i$$

Suppose that this is proven out to a particular level n. Then I have:

$$\left[\hat{A},.(n \quad commutators).,\left[\hat{A},\hat{B}\right]\right] = \sum_{i=0}^{n}\binom{n}{i}(-1)^i\hat{A}^{n-i}\hat{B}\hat{A}^i$$

Then, for row n + 1, I have:

$$\left[\hat{A},.(n+1 \quad commutators).,\left[\hat{A},\hat{B}\right]\right]$$
$$= \hat{A}\left[\hat{A},.(n \quad commutators).,\left[\hat{A},\hat{B}\right]\right] - \left[\hat{A},.(n \quad commutators).,\left[\hat{A},\hat{B}\right]\right]\hat{A}$$
$$= \hat{A}\left[\sum_{i=0}^{n}\binom{n}{i}(-1)^i \hat{A}^{n-i}\hat{B}\hat{A}^i\right] - \left[\sum_{i=0}^{n}\binom{n}{i}(-1)^i \hat{A}^{n-i}\hat{B}\hat{A}^i\right]\hat{A}$$
$$= \left[\sum_{i=0}^{n}\binom{n}{i}(-1)^i \hat{A}^{n+1-i}\hat{B}\hat{A}^i\right] - \left[\sum_{i=0}^{n}\binom{n}{i}(-1)^i \hat{A}^{n-i}\hat{B}\hat{A}^{i+1}\right]$$
$$define \quad \binom{n}{-1},\binom{n}{n+1} = 0, re-index:$$
$$= \left[\sum_{i=0}^{n}\binom{n}{i}(-1)^i \hat{A}^{n+1-i}\hat{B}\hat{A}^i\right] - \left[\sum_{i=1}^{n+1}\binom{n}{i-1}(-1)^{(i-1)} \hat{A}^{n-(i-1)}\hat{B}\hat{A}^{i}\right]$$
$$= \left[\sum_{i=0}^{n}\binom{n}{i}(-1)^i \hat{A}^{n+1-i}\hat{B}\hat{A}^i\right] + \left[\sum_{i=1}^{n+1}\binom{n}{i-1}(-1)^{i} \hat{A}^{n-(i-1)}\hat{B}\hat{A}^{i}\right]$$
$$= \sum_{i=0}^{n+1}(-1)^i\left[\binom{n}{i}+\binom{n}{i-1}\right]\hat{A}^{n+1-i}\hat{B}\hat{A}^i$$
$$= \sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}\hat{A}^{n+1-i}\hat{B}\hat{A}^i$$

I have used a well-known identity on combinations, $\binom{n}{i}+\binom{n}{i-1}=\binom{n+1}{i}$.

b) Consider operators $\hat{A},\hat{B}$ whose commutator is a complex number $\left[\hat{A},\hat{B}\right]=C$.

Show that $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}C}$.

Write: $U(\alpha)=e^{\alpha\hat{A}}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})}$. Now:

$$\frac{\partial U(\alpha)}{\partial\alpha} = \hat{A}e^{\alpha\hat{A}}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})} + e^{\alpha\hat{A}}\hat{B}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})} - e^{\alpha\hat{A}}e^{\alpha\hat{B}}\left(\hat{A}+\hat{B}\right)e^{-\alpha(\hat{A}+\hat{B})}$$
$$= \hat{A}e^{\alpha\hat{A}}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})} + e^{\alpha\hat{A}}\left[\hat{B}e^{\alpha\hat{B}}\right]e^{-\alpha(\hat{A}+\hat{B})} - e^{\alpha\hat{A}}e^{\alpha\hat{B}}\hat{A}e^{-\alpha(\hat{A}+\hat{B})} - e^{\alpha\hat{A}}\left[e^{\alpha\hat{B}}\hat{B}\right]e^{-\alpha(\hat{A}+\hat{B})}$$
$$= \hat{A}e^{\alpha\hat{A}}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})} + \left[e^{\alpha\hat{A}}\hat{B}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})}\right] - e^{\alpha\hat{A}}e^{\alpha\hat{B}}\hat{A}e^{-\alpha(\hat{A}+\hat{B})} - \left[e^{\alpha\hat{A}}\hat{B}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})}\right]$$
$$= \hat{A}e^{\alpha\hat{A}}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})} - e^{\alpha\hat{A}}e^{\alpha\hat{B}}\hat{A}e^{-\alpha(\hat{A}+\hat{B})}$$
$$= \hat{A}e^{\alpha\hat{A}}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})} - \hat{A}e^{\alpha\hat{A}}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})} - e^{\alpha\hat{A}}\left[e^{\alpha\hat{B}},\hat{A}\right]e^{-\alpha(\hat{A}+\hat{B})}$$
$$= -e^{\alpha\hat{A}}\left[e^{\alpha\hat{B}},\hat{A}\right]e^{-\alpha(\hat{A}+\hat{B})}$$

Briefly consider

$$\left[e^{\alpha\hat{B}},\hat{A}\right]=\left[\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\hat{B}^n,\hat{A}\right]=\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\left[\hat{B}^n,\hat{A}\right]$$

Now I need:

$$\left[\hat{B}^n,\hat{A}\right]$$

$$recurse \quad on \quad \left[\hat{B}^n,\hat{A}\right]$$

$$\left[\hat{B},\hat{A}\right]=-C$$

$$\left[\hat{B}^n,\hat{A}\right]=\hat{B}^{n-1}\left[\hat{B},\hat{A}\right]+\left[\hat{B}^{n-1},\hat{A}\right]\hat{B}=\left\{\left[\hat{B}^{n-1},\left[\hat{B},\hat{A}\right]\right]\right\}_{\to 0}+\left[\hat{B},\hat{A}\right]\hat{B}^{n-1}+\left[\hat{B}^{n-1},\hat{A}\right]\hat{B}$$

Since the bracketed part goes to zero (since C is just a complex number), I have

$$\left[\hat{B}^1,\hat{A}\right]=-C$$

$$\left[\hat{B}^n,\hat{A}\right]=\hat{B}^{n-1}\left[\hat{B},\hat{A}\right]+\left[\hat{B}^{n-1},\hat{A}\right]\hat{B}=\left\{\left[\hat{B}^{n-1},\left[\hat{B},\hat{A}\right]\right]\right\}_{\to 0}-C\hat{B}^{n-1}+\left[\hat{B}^{n-1},\hat{A}\right]\hat{B}$$

$$=-nC\hat{B}^{n-1}$$

It was necessary here to invoke $\left[\hat{B},\left[\hat{B},\hat{A}\right]\right]=0$ since $\left[\hat{A},\hat{B}\right]=C$ complex, otherwise the bracketed part would not necessarily have vanished. The remaining commutator on the right side provides n -1 more contributions of $C\hat{B}^{n-1}$.

Continuing, I have

$$\left[e^{\alpha\hat{B}},\hat{A}\right]=\left[\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\hat{B}^n,\hat{A}\right]=\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\left[\hat{B}^n,\hat{A}\right]$$

$$=-C\sum_{n=1}^{\infty}\frac{\alpha^n}{n!}n\hat{B}^{n-1}=-\alpha C\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}n\hat{B}^n=-\alpha Ce^{\alpha\hat{B}}$$

$$\frac{\partial U(\alpha)}{\partial\alpha}=\alpha Ce^{\alpha\hat{A}}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})}=\alpha CU(\alpha)$$

$$Solving,$$

$$U(\alpha)=e^{\frac{\alpha^2}{2}C}$$

$$Giving:$$

$$U(\alpha)=e^{\alpha\hat{A}}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})}=e^{\frac{\alpha^2}{2}C}$$

$$U(\alpha)=e^{\alpha\hat{A}}e^{\alpha\hat{B}}e^{-\alpha(\hat{A}+\hat{B})}=e^{-\frac{\alpha^2}{2}C}$$

$$e^{\alpha\hat{A}}e^{\alpha\hat{B}}e^{-\frac{\alpha^2}{2}C}=e^{\alpha(\hat{A}+\hat{B})}$$

All that's left to do is to set $\alpha=1$ for $e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}C}=e^{(\hat{A}+\hat{B})}$.

2) For a simple harmonic oscillator, $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(a^+a + \frac{1}{2}\right)$**. The ground state is** $|0\rangle$ **so that** $\hat{a}|0\rangle = 0$**, and a coherent state** $|\alpha\rangle$ **is an eigenstate of** $\hat{a}$**:** $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$**.**

a) Show that $|\alpha\rangle = e^{(\alpha\hat{a}^+ - \alpha^*\hat{a})}|0\rangle$

Using the Baker-Haussdorf relation proven in part 1, I have:

$$\hat{a}|\alpha\rangle = \hat{a}e^{(\alpha\hat{a}^+ - \alpha^*\hat{a})}|0\rangle$$
$$lemma: [\alpha\hat{a}^+, -\alpha^*\hat{a}] = |\alpha|^2[\hat{a},\hat{a}^+] = |\alpha|^2$$
$$\hat{a}e^{(\alpha\hat{a}^+ - \alpha^*\hat{a})}|0\rangle = \hat{a}e^{\alpha\hat{a}^+}e^{-\alpha^*\hat{a}}e^{-\frac{|a^2|}{2}}|0\rangle$$

Now I make use of the identity derived earlier

$$\left[\hat{A}, e^{\alpha\hat{B}}\right] = \alpha\left[\hat{A},\hat{B}\right]e^{\alpha\hat{B}}$$
$$\hat{A} \to \hat{a} \quad \hat{B} \to \hat{a}^+$$
$$\left[\hat{a}, e^{\alpha\hat{a}^+}\right] = \alpha e^{\alpha\hat{a}^+}$$

and certainly, $\hat{a}$ commutes with the rest of the expression $\left[\hat{a}, e^{-\alpha^*\hat{a}}e^{-\frac{|a^2|}{2}}\right] = 0$

$$\hat{a}e^{\alpha\hat{a}^+}e^{-\alpha^*\hat{a}}e^{-\frac{|a^2|}{2}}|0\rangle = \alpha e^{\alpha\hat{a}^+}e^{-\alpha^*\hat{a}}e^{-\frac{|a^2|}{2}}|0\rangle = \alpha e^{(\alpha\hat{a}^+ - \alpha^*\hat{a})}|0\rangle = \alpha|\alpha\rangle.$$

b) Use the Baker-Haussdorf relation to show that $D(\alpha) = e^{\alpha\hat{a}^+ - a^*\hat{a}} = e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^+}e^{-a^*\hat{a}}$ **and that** $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$ **with** $\hat{a}^+\hat{a}|n\rangle = \hat{n}|n\rangle = n|n\rangle$

As in part a,

$$\left|\alpha\right\rangle = e^{\left(\alpha\hat{a}^{+} - \alpha^{*}\hat{a}\right)}\left|0\right\rangle$$

$$lemma : \left[\alpha\hat{a}^{+}, -\alpha^{*}\hat{a}\right] = \left|\alpha\right|^{2}\left[\hat{a}, \hat{a}^{+}\right] = \left|\alpha\right|^{2}$$

$$e^{\left(\alpha\hat{a}^{+} - \alpha^{*}\hat{a}\right)}\left|0\right\rangle = e^{\alpha\hat{a}^{+}} e^{-\alpha^{*}\hat{a}} e^{-\frac{\left|\alpha^{2}\right|}{2}}\left|0\right\rangle = e^{-\frac{\left|\alpha^{2}\right|}{2}} e^{\alpha\hat{a}^{+}} e^{-\alpha^{*}\hat{a}}\left|0\right\rangle$$

$$e^{-\alpha^{*}\hat{a}}\left|0\right\rangle = \left(1 - \alpha^{*}\hat{a} + \ldots\right)\left|0\right\rangle = \left|0\right\rangle$$

$$e^{-\frac{\left|\alpha^{2}\right|}{2}} e^{\alpha\hat{a}^{+}}\left|0\right\rangle = e^{-\frac{\left|\alpha^{2}\right|}{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{n!}\left(\hat{a}^{+}\right)^{n}\left|0\right\rangle = e^{-\frac{\left|\alpha^{2}\right|}{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{n!}\sqrt{n!}\left|n\right\rangle = e^{-\frac{\left|\alpha^{2}\right|}{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}\left|n\right\rangle$$

since $\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \quad \hat{a}^{+}\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$.

c) Compute $\left\langle\alpha\right|\hat{x}\left|\alpha\right\rangle$ and $\left\langle\alpha\right|\hat{p}\left|\alpha\right\rangle$.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^{+} + \hat{a}\right) \quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^{+} - \hat{a}\right)$$

$$\left\langle\alpha\right|\hat{x}\left|\alpha\right\rangle = \sqrt{\frac{\hbar}{2m\omega}} e^{-\left|\alpha\right|^{2}}\sum_{n,m=0}^{\infty}\left\langle m\right|\frac{\left(\alpha^{m}\right)^{*}}{\sqrt{m!}}\left(\hat{a}^{+} + \hat{a}\right)\frac{\alpha^{n}}{\sqrt{n!}}\left|n\right\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} e^{-\left|\alpha\right|^{2}}\left[\sum_{n,m=0}^{\infty}\left\langle m\right|\frac{\left(\alpha^{m}\right)^{*}}{\sqrt{m!}}\hat{a}^{+}\frac{\alpha^{n}}{\sqrt{n!}}\left|n\right\rangle + \sum_{n,m=0}^{\infty}\left\langle m\right|\frac{\left(\alpha^{m}\right)^{*}}{\sqrt{m!}}\hat{a}\frac{\alpha^{n}}{\sqrt{n!}}\left|n\right\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} e^{-\left|\alpha\right|^{2}}\left[\sum_{n,m=0}^{\infty}\left\langle m\right|\frac{\left(\alpha^{m}\right)^{*}}{\sqrt{m!}}\frac{\alpha^{n}}{\sqrt{n!}}\sqrt{n+1}\left|n+1\right\rangle + \sum_{m=0,n=1}^{\infty}\left\langle m\right|\frac{\left(\alpha^{m}\right)^{*}}{\sqrt{m!}}\sqrt{n}\frac{\alpha^{n}}{\sqrt{n!}}\left|n-1\right\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} e^{-\left|\alpha\right|^{2}}\left[\sum_{m=0}^{\infty}\frac{\left(\alpha^{m}\right)^{*}}{\sqrt{m!}}\frac{\alpha^{m-1}}{\sqrt{(m-1)!}}\sqrt{m} + \sum_{m=0}^{\infty}\frac{\left(\alpha^{m}\right)^{*}}{\sqrt{m!}}\sqrt{m+1}\frac{\alpha^{m+1}}{\sqrt{m+1!}}\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} e^{-\left|\alpha\right|^{2}}\left[\sum_{m=0}^{\infty}\frac{\left|\alpha\right|^{2(m-1)}}{(m-1)!}\alpha^{*} + \sum_{m=0}^{\infty}\frac{\left|\alpha\right|^{2m}}{m!}\alpha\right] = \sqrt{\frac{\hbar}{2m\omega}} e^{-\left|\alpha\right|^{2}}\left[\left(\alpha^{*} + \alpha\right)e^{\left|\alpha\right|^{2}}\right] = \left(\alpha^{*} + \alpha\right)\sqrt{\frac{\hbar}{2m\omega}}$$

$$\langle\alpha|\hat{p}|\alpha\rangle = i\sqrt{\frac{\hbar m\omega}{2}}e^{-|\alpha|^2}\sum_{n,m=0}^{\infty}\langle m|\frac{(\alpha^m)^*}{\sqrt{m!}}(\hat{a}^+ - \hat{a})\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}e^{-|\alpha|^2}\left[\sum_{n,m=0}^{\infty}\langle m|\frac{(\alpha^m)^*}{\sqrt{m!}}\hat{a}^+\frac{\alpha^n}{\sqrt{n!}}|n\rangle - \sum_{n,m=0}^{\infty}\langle m|\frac{(\alpha^m)^*}{\sqrt{m!}}\hat{a}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\right]$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}e^{-|\alpha|^2}\left[\sum_{n,m=0}^{\infty}\langle m|\frac{(\alpha^m)^*}{\sqrt{m!}}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n+1}|n+1\rangle - \sum_{m=0,n=1}^{\infty}\langle m|\frac{(\alpha^m)^*}{\sqrt{m!}}\sqrt{n}\frac{\alpha^n}{\sqrt{n!}}|n-1\rangle\right]$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}e^{-|\alpha|^2}\left[\sum_{m=0}^{\infty}\frac{(\alpha^m)^*}{\sqrt{m!}}\frac{\alpha^{m-1}}{\sqrt{(m-1)!}}\sqrt{m} - \sum_{m=0}^{\infty}\frac{(\alpha^m)^*}{\sqrt{m!}}\sqrt{m+1}\frac{\alpha^{m+1}}{\sqrt{m+1!}}\right]$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}e^{-|\alpha|^2}\left[\sum_{m=0}^{\infty}\frac{|\alpha|^{2(m-1)}}{(m-1)!}\alpha^* - \sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{m!}\alpha\right] = i\sqrt{\frac{\hbar m\omega}{2}}e^{-|\alpha|^2}\left[(\alpha^* - \alpha)e^{|\alpha|^2}\right] = i(\alpha^* - \alpha)\sqrt{\frac{\hbar m\omega}{2}}$$

3) Consider a simple harmonic oscillator alongside one that has been displaced by an external force F:

$$H_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$H_F = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 - F\hat{x}$$

a) Show that the ground state of H_F is a coherent state in the eigenbasis of H_0.

$$H_F = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\left[\hat{x}^2 - \frac{2F}{m\omega^2}\hat{x}\right] = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\left(\hat{x} - \frac{F}{m\omega^2}\right)^2 - \frac{F^2}{2m\omega^2}$$

$$Define \quad x_0 = \frac{F}{m\omega^2}$$

$$H_F = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(\hat{x} - x_0)^2 - \frac{1}{2}m\omega^2x_0^2$$

In order to find the ground state eigenfunction exactly, I first need to solve the differential equation:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \quad \hat{x} = x$$

$$H_F = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2(x - x_0)^2 - \frac{1}{2}m\omega^2x_0^2$$

I may define a change of variables:

$$y = x - x_0 \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial x}$$

$$H_F = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m\omega^2 y^2 - \frac{1}{2}m\omega^2 x_0^2$$

I recall that the ground-state eigenfunction of a Harmonic Oscillator (the zero-point energy term is irrelevant) is a Gaussian: Namely, take the ansatz, sans a normalization factor, is $\psi_{F,0}(y) = e^{-\alpha y^2}$.

$$H_F\psi_{F,0}(y) = -\frac{\hbar^2}{2m}\left[-2\alpha + 4\alpha^2\right]\psi_{F,0}(y) + \frac{1}{2}m\omega^2 y^2\psi_{F,0}(y) - \frac{1}{2}m\omega^2 x_0^2\psi_{F,0}(y)$$

$$\therefore \frac{4\hbar^2}{2m}\alpha^2 = \frac{1}{2}m\omega^2 \quad \alpha = \frac{m\omega}{2\hbar}$$

$$H_F\psi_{F,0}(y) = \left[\frac{\hbar\omega}{2} - \frac{1}{2}m\omega^2 x_0^2\right]\psi_0(y)$$

It is theoretically possible to project this eigenfunction in the y coordinates into the space of x coordinates, but is not necessary. The only necessity to be a coherent state of the unperturbed harmonic oscillator Hamiltonian is that the perturbed ground state is an eigenfunction of the unperturbed lowering operator of the unperturbed harmonic oscillator. I then write:

$$\psi_{F,0}(x) = e^{-\frac{m\omega}{2\hbar}(x-x_0)^2}$$

The lowering operator $a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)$

So that

$$a\psi_{F,0}(x) = \sqrt{\frac{m\omega}{2\hbar}}x\psi_{F,0}(x) + \sqrt{\frac{m\omega}{2\hbar}}\cdot\frac{\hbar}{m\omega}\left[-2\cdot\frac{m\omega}{2\hbar}(x - x_0)\right]\psi_{F,0}(x)$$

$$= \sqrt{\frac{m\omega}{2\hbar}}x\psi_{F,0}(x) - \sqrt{\frac{m\omega}{2\hbar}}(x - x_0)\psi_{F,0}(x)$$

$$= \sqrt{\frac{m\omega}{2\hbar}}x_0\psi_{F,0}(x)$$

This indicates that the ground state of the perturbed Hamiltonian is a coherent state of the unperturbed Hamiltonian with eigenvalue $\sqrt{\frac{m\omega}{2\hbar}}x_0 = \sqrt{\frac{m\omega}{2\hbar}}\left[\frac{F}{m\omega^2}\right]$ (in any basis).

Show that $H_F = e^{i\frac{\hat{p}}{\hbar}x_0}H_0 e^{-i\frac{\hat{p}}{\hbar}x_0}$**. What is** x_0 **as a function of F?**

Use the identity shown in problem 1a.

In this case, the series should truncate:

$$\left[i\frac{\hat{p}}{\hbar}x_0, H_0\right] = \frac{ix_0}{\hbar}\left[\hat{p}, H_0\right] = \frac{ix_0}{\hbar}\left[\hat{p}, \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right] = \frac{ix_0 m\omega^2}{2\hbar}\left[\hat{p}, \hat{x}^2\right]$$

$$use \quad \left[\hat{p}, \hat{x}\right] = -i\hbar$$

$$= \frac{ix_0 m\omega^2}{2\hbar}\left[\hat{p}, \hat{x}^2\right] = \frac{ix_0 m\omega^2}{2\hbar}\left(\left[\hat{p}, \hat{x}\right]\hat{x} + \hat{x}\left[\hat{p}, \hat{x}\right]\right) = x_0 m\omega^2\hat{x}$$

$$\left[i\frac{\hat{p}}{\hbar}x_0, \left[i\frac{\hat{p}}{\hbar}x_0, H_0\right]\right] = \frac{ix_0}{\hbar}\left[\hat{p}, x_0 m\omega^2\hat{x}\right] = \frac{im\omega^2 x_0{}^2}{\hbar}\left[\hat{p}, \hat{x}\right] = -m\omega^2 x_0{}^2$$

$$\left[i\frac{\hat{p}}{\hbar}x_0, \left[i\frac{\hat{p}}{\hbar}x_0, \left[i\frac{\hat{p}}{\hbar}x_0, H_0\right]\right]\right] = \left[i\frac{\hat{p}}{\hbar}x_0, -m\omega^2 x_0{}^2\right] = 0, etc.$$

Now from substitution into the solution from 1a, I have that

$$H_F = e^{i\frac{\hat{p}}{\hbar}x_0} H_0 e^{-i\frac{\hat{p}}{\hbar}x_0} = H_0 + x_0 m\omega^2\hat{x} - \frac{1}{2}m\omega^2 x_0^2$$

(this includes a zero-point energy contribution that could be ignored.)

Then in this case, $F = x_0 m\omega^2 \quad \therefore \quad x_0 = \dfrac{F}{m\omega^2}$.

4) **Consider three free indistinguishable particles, construct the fermionic and bosonic wave functions. Do not include spin.**

The free-particle wave function is $\psi_n(\vec{x}) = e^{\frac{i}{\hbar}\vec{p}_n\cdot\vec{x}}$ for a particle n with momentum $\vec{p}_n$.

Fermionic:

$$\begin{vmatrix} \psi_a(\vec{r}_1) & \psi_a(\vec{r}_2) & \psi_a(\vec{r}_3) \\ \psi_b(\vec{r}_1) & \psi_b(\vec{r}_2) & \psi_b(\vec{r}_3) \\ \psi_c(\vec{r}_1) & \psi_c(\vec{r}_2) & \psi_c(\vec{r}_3) \end{vmatrix}$$

$$= \frac{1}{\sqrt{6}}\begin{bmatrix} \psi_a(\vec{r}_1)\psi_b(\vec{r}_2)\psi_c(\vec{r}_3) - \psi_a(\vec{r}_1)\psi_c(\vec{r}_3)\psi_b(\vec{r}_2) - \psi_a(\vec{r}_2)\psi_b(\vec{r}_3)\psi_c(\vec{r}_1) \\ + \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\psi_c(\vec{r}_3) + \psi_a(\vec{r}_3)\psi_b(\vec{r}_1)\psi_c(\vec{r}_2) - \psi_a(\vec{r}_3)\psi_b(\vec{r}_2)\psi_c(\vec{r}_1) \end{bmatrix}$$

Bosonic:

$$= \frac{1}{\sqrt{27}}\left(\psi_a(\vec{r}_1) + \psi_b(\vec{r}_1) + \psi_c(\vec{r}_1)\right)\left(\psi_a(\vec{r}_2) + \psi_b(\vec{r}_2) + \psi_c(\vec{r}_2)\right)\left(\psi_a(\vec{r}_3) + \psi_b(\vec{r}_3) + \psi_c(\vec{r}_3)\right)$$