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Quantum Mechanics 3 Final

1) Consider a homogeneous, weakly interacting Bose gas with Hamiltonian

$$\hat{H} = \int d^3x \hat{\psi}^\dagger(\vec{x}) \left(-\frac{\hbar^2 \nabla^2}{2m} \right) \hat{\psi}(\vec{x}) + \iint d^3x d^3y \hat{\psi}^\dagger(\vec{x}) \hat{\psi}^\dagger(\vec{y}) V(\vec{x} - \vec{y}) \hat{\psi}(\vec{y}) \hat{\psi}(\vec{x}).$$

a) Consider $V(\vec{x} - \vec{y}) = -|V_0| \delta^3(\vec{x} - \vec{y})$ and assume a condensate with N_0 particles. Obtain the operator $\hat{K} = \hat{H} - \mu \hat{N}$ in the Bogoliubov approximation. By a canonical Bogoliubov transformation bring it to the form $\hat{K} = \sum_k \hbar \Omega(k) \hat{c}_k^\dagger \hat{c}_k + K_0$. Show that $\Omega(k)$ becomes imaginary for some values of $0 < k < k_{\max}$.

Let $\hat{\psi}(\vec{x}) = \sum_n U_n(\vec{x}) \hat{a}_n$ be normalized so that $\int d^3x U_n^*(\vec{x}) U_m(\vec{x}) = \delta_{m,n}$.

Further, let $\left(-\frac{\hbar^2 \nabla^2}{2m} \right) U_n(\vec{x}) = \varepsilon(n) U_n(\vec{x})$ and $\hat{N} = \int d^3x \hat{\psi}^\dagger(\vec{x}) \hat{\psi}(\vec{x})$.

Now I have:

$$\int d^3x \hat{\psi}^\dagger(\vec{x}) \left(-\frac{\hbar^2 \nabla^2}{2m} \right) \hat{\psi}(\vec{x}) = \sum_k \varepsilon(k) \hat{a}_k^\dagger \hat{a}_k.$$

$$\hat{N} = \int d^3x \hat{\psi}^\dagger(\vec{x}) \hat{\psi}(\vec{x}) = \sum_k \hat{a}_k^\dagger \hat{a}_k.$$

Next, I consider:

$$\begin{aligned} & \iint d^3x d^3y \hat{\psi}^\dagger(\vec{x}) \hat{\psi}^\dagger(\vec{y}) V(\vec{x} - \vec{y}) \hat{\psi}(\vec{y}) \hat{\psi}(\vec{x}) \\ &= -\frac{|V_0|}{V} \int d^3x \hat{\psi}^\dagger(\vec{x}) \hat{\psi}^\dagger(\vec{x}) \hat{\psi}(\vec{x}) \hat{\psi}(\vec{x}) \\ &= -\frac{|V_0|}{V} \sum_{i,j,k,l} \int d^3x U_i^*(\vec{x}) U_j^*(\vec{x}) U_k(\vec{x}) U_l(\vec{x}) \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l \end{aligned}$$

For a weakly interacting gas, take essentially free particles which travel uniformly in spatial directions: $U_n(\vec{x}) \propto e^{im|\vec{x}|}$. Then, I see that

$$-\frac{|V_0|}{V} \sum_{i,j,k,l} \int d^3x U_i^*(\vec{x}) U_j^*(\vec{x}) U_k(\vec{x}) U_l(\vec{x}) \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l$$

has $i + j + k + l = 0$ in order to survive, since $\int dx e^{ikx} = \delta_{k,0}$, so

$$\int \int d^3x d^3y \hat{\psi}^+(\bar{x}) \hat{\psi}^+(\bar{y}) V(\bar{x} - \bar{y}) \hat{\psi}(\bar{y}) \hat{\psi}(\bar{x}) = -\frac{|V_0|}{V} \sum_{q,k,l} \hat{a}_{l+q}^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_l.$$

So, I may combine these pieces and rewrite:

$$\hat{K} = \hat{H} - \mu \hat{N} = \sum_k [\varepsilon(k) - \mu] \hat{a}_k^+ \hat{a}_k - \frac{|V_0|}{V} \sum_{q,k,l} \hat{a}_{l+q}^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_l.$$

In the canonical Bogoliubov approximation, I take:

$$\hat{a}_0 = \sqrt{N_0} e^{i\theta} + \hat{b}_0 \quad \hat{a}_{k \neq 0} = \hat{b}_{k \neq 0}$$

Expanding to highlight zero terms versus nonzero terms a bit, then, I have:

$$\begin{aligned} \hat{K} &= \sum_k [\varepsilon(k) - \mu] \hat{a}_k^+ \hat{a}_k - \frac{|V_0|}{V} \sum_{q,k,l} \hat{a}_{l+q}^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_l \\ &= \varepsilon(0) \hat{a}_0^+ \hat{a}_0 + \sum_{k \neq 0} \varepsilon(k) \hat{a}_k^+ \hat{a}_k \\ &\quad - \frac{|V_0|}{V} \sum_q \hat{a}_0^+ \hat{a}_0^+ \hat{a}_q \hat{a}_{-q} - \frac{|V_0|}{V} \sum_{q \neq 0} \hat{a}_0^+ \hat{a}_{k-q}^+ \hat{a}_0 \hat{a}_{-q} - \frac{|V_0|}{V} \sum_{k \neq q, k \neq 0} \hat{a}_0^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_{-q} \\ &\quad - \frac{|V_0|}{V} \sum_{q \neq 0} \hat{a}_q^+ \hat{a}_0^+ \hat{a}_q \hat{a}_0 - \frac{|V_0|}{V} \sum_{q \neq 0} \hat{a}_q^+ \hat{a}_{-q}^+ \hat{a}_0 \hat{a}_0 - \frac{|V_0|}{V} \sum_{k \neq q, k \neq 0} \hat{a}_q^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_0 \\ &\quad - \frac{|V_0|}{V} \sum_q \hat{a}_{l+q}^+ \hat{a}_0^+ \hat{a}_q \hat{a}_l - \frac{|V_0|}{V} \sum_{q \neq 0} \hat{a}_{l+q}^+ \hat{a}_{-q}^+ \hat{a}_0 \hat{a}_l - \frac{|V_0|}{V} \sum_{l \neq -q, l \neq 0, k \neq q, k \neq 0} \hat{a}_{l+q}^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_l \end{aligned}$$

Splitting these, then, where necessary into $q = 0$ and $q \neq 0$ cases, I have:

$$\hat{K} = (\text{one - particle energies}): [\varepsilon(0) - \mu] \hat{a}_0^+ \hat{a}_0 + \sum_{k \neq 0} [\varepsilon(k) - \mu] \hat{a}_k^+ \hat{a}_k$$

$$(0 \text{ excited interaction}): -\frac{|V_0|}{V} \hat{a}_0^+ \hat{a}_0^+ \hat{a}_0 \hat{a}_0$$

$$(2 \text{ excited interaction}): -\frac{|V_0|}{V} \sum_{q \neq 0} \hat{a}_0^+ \hat{a}_0^+ \hat{a}_q \hat{a}_{-q} - \frac{|V_0|}{V} \sum_{q \neq 0} \hat{a}_0^+ \hat{a}_{-q}^+ \hat{a}_0 \hat{a}_{-q} - \frac{|V_0|}{V} \sum_{k \neq 0} \hat{a}_0^+ \hat{a}_k^+ \hat{a}_k \hat{a}_0$$

$$- \frac{|V_0|}{V} \sum_{q \neq 0} \hat{a}_q^+ \hat{a}_0^+ \hat{a}_q \hat{a}_0 - \frac{|V_0|}{V} \sum_{q \neq 0} \hat{a}_q^+ \hat{a}_{-q}^+ \hat{a}_0 \hat{a}_0 - \frac{|V_0|}{V} \sum_{l \neq 0} \hat{a}_l^+ \hat{a}_0^+ \hat{a}_0 \hat{a}_l$$

$$(3 \text{ excited interaction}): -\frac{|V_0|}{V} \sum_{k \neq q, k \neq 0} \hat{a}_0^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_{-q} - \frac{|V_0|}{V} \sum_{k \neq q, k \neq 0} \hat{a}_q^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_0$$

$$- \frac{|V_0|}{V} \sum_{q \neq 0, l \neq -q, l \neq 0} \hat{a}_{l+q}^+ \hat{a}_0^+ \hat{a}_q \hat{a}_l - \frac{|V_0|}{V} \sum_{q \neq 0, l \neq -q, l \neq 0} \hat{a}_{l+q}^+ \hat{a}_{-q}^+ \hat{a}_0 \hat{a}_l$$

$$(4 \text{ excited interaction}): -\frac{|V_0|}{V} \sum_{l \neq -q, l \neq 0, k \neq q, k \neq 0} \hat{a}_{l+q}^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_l$$

The Feynman diagrams of these show the various processes of scattering, including pair production and creation from the condensate. Now switching over to the \hat{b} language, and considering only the interaction terms:

$$\hat{K}_I =$$

$$(0 \text{ excited interaction}): -\frac{|V_0|}{V} (\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+) (\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+) (\sqrt{N_0} e^{i\theta} + \hat{b}_0) (\sqrt{N_0} e^{i\theta} + \hat{b}_0)$$

$$(2 \text{ excited interaction}): -\frac{|V_0|}{V} \sum_{q \neq 0} (\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+) (\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+) \hat{b}_q \hat{b}_{-q}$$

$$- \frac{|V_0|}{V} \sum_{q \neq 0} (\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+) \hat{b}_{-q}^+ (\sqrt{N_0} e^{i\theta} + \hat{b}_0) \hat{b}_{-q} - \frac{|V_0|}{V} \sum_{k \neq 0} (\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+) \hat{b}_k^+ \hat{b}_k (\sqrt{N_0} e^{i\theta} + \hat{b}_0)$$

$$- \frac{|V_0|}{V} \sum_{q \neq 0} \hat{b}_q^+ (\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+) \hat{b}_q (\sqrt{N_0} e^{i\theta} + \hat{b}_0) - \frac{|V_0|}{V} \sum_{q \neq 0} \hat{b}_q^+ \hat{b}_{-q}^+ (\sqrt{N_0} e^{i\theta} + \hat{b}_0) (\sqrt{N_0} e^{i\theta} + \hat{b}_0)$$

$$- \frac{|V_0|}{V} \sum_{l \neq -q, l \neq 0} \hat{b}_l^+ (\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+) (\sqrt{N_0} e^{i\theta} + \hat{b}_0) \hat{b}_l$$

$$(3 \text{ excited interaction}): -\frac{|V_0|}{V} \sum_{k \neq q, k \neq 0} (\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+) \hat{b}_{k-q}^+ \hat{b}_k \hat{b}_{-q} - \frac{|V_0|}{V} \sum_{k \neq q, k \neq 0} \hat{b}_q^+ \hat{b}_{k-q}^+ \hat{b}_k (\sqrt{N_0} e^{i\theta} + \hat{b}_0)$$

$$- \frac{|V_0|}{V} \sum_{l \neq -q, l \neq 0} \hat{b}_{l+q}^+ (\sqrt{N_0} e^{-i\theta} + \hat{b}_0^+) \hat{b}_q \hat{b}_l - \frac{|V_0|}{V} \sum_{l \neq -q, l \neq 0} \hat{b}_{l+q}^+ \hat{b}_{-q}^+ (\sqrt{N_0} e^{i\theta} + \hat{b}_0) \hat{b}_l$$

$$(4 \text{ excited interaction}): -\frac{|V_0|}{V} \sum_{l \neq -q, l \neq 0, k \neq q, k \neq 0} \hat{b}_{l+q}^+ \hat{b}_{k-q}^+ \hat{b}_k \hat{b}_l$$

Rather than explicitly expanding these in all of their terrible glory, I note a pattern: the only way to pick up a factor of $\sqrt{N_0}$ in the above is to sacrifice the presence of a \hat{b}_0 . In this way, then, I see that for each factor of $\sqrt{N_0}$ that appears, the order of the term is reduced by one. Then, terms up to quadratic in \hat{b}_0 have coefficients $N_0^2, N_0\sqrt{N_0}, N_0$ and higher order terms have coefficients $\sqrt{N_0}, 1$. Then, dropping the latter-order terms, I get:

(0 *excited interaction*) \rightarrow

$$-\frac{|V_0|}{V} \left[N_0^2 + 2N_0\sqrt{N_0}e^{i\theta}\hat{b}_0^+ + N_0e^{2i\theta}\hat{b}_0^+\hat{b}_0^+ + 2N_0\sqrt{N_0}e^{-i\theta}\hat{b}_0 + 4N_0\hat{b}_0^+\hat{b}_0 + N_0e^{-2i\theta}\hat{b}_0\hat{b}_0 + \theta(\sqrt{N_0}) + \theta(1) \right]$$

(2 *excited interaction*) \rightarrow

$$\begin{aligned} & -\frac{|V_0|}{V} \sum_{q \neq 0} N_0 e^{-2i\theta} \hat{b}_q^+ \hat{b}_{-q} - \frac{|V_0|}{V} \sum_{q \neq 0} N_0 \hat{b}_{-q}^+ \hat{b}_{-q} - \frac{|V_0|}{V} \sum_{k \neq 0} N_0 \hat{b}_k^+ \hat{b}_k \\ & -\frac{|V_0|}{V} \sum_{q \neq 0} N_0 \hat{b}_q^+ \hat{b}_q - \frac{|V_0|}{V} \sum_{q \neq 0} N_0 e^{2i\theta} \hat{b}_q^+ \hat{b}_{-q}^+ - \frac{|V_0|}{V} \sum_{l \neq 0} N_0 \hat{b}_l^+ \hat{b}_l + \theta(\sqrt{N_0}) \\ & \rightarrow -\frac{|V_0|}{V} N_0 \left[4 \sum_{q \neq 0} \hat{b}_q^+ \hat{b}_q + e^{2i\theta} \sum_{q \neq 0} \hat{b}_q^+ \hat{b}_{-q}^+ + e^{-2i\theta} \sum_{q \neq 0} \hat{b}_q \hat{b}_{-q} \right] \end{aligned}$$

(3 *excited interaction*) \rightarrow

$$+ \theta(\sqrt{N_0}) + \theta(1)$$

(4 *excited interaction*): \rightarrow

$$+ \theta(1)$$

The next step is to eliminate the terms linear in \hat{b}, \hat{b}^+ . This is done through the elimination of \hat{K}_{cl} through the Gross-Ginzburg-Pitaevskii equation $\frac{\partial \hat{K}_{cl}}{\partial N_0} = 0$. Let me

first write my expression for \hat{K} :

$$\begin{aligned} \hat{K} &= [\varepsilon(0) - \mu] \left(N_0 + \sqrt{N_0} e^{-i\theta} \hat{b}_0 + \sqrt{N_0} e^{i\theta} \hat{b}_0^+ + \hat{b}_0^+ \hat{b}_0 \right) + \sum_{k \neq 0} [\varepsilon(k) - \mu] \hat{b}_k^+ \hat{b}_k \\ & - \frac{|V_0|}{V} \left[N_0^2 + 2N_0\sqrt{N_0}e^{i\theta}\hat{b}_0^+ + N_0e^{2i\theta}\hat{b}_0^+\hat{b}_0^+ + 2N_0\sqrt{N_0}e^{-i\theta}\hat{b}_0 + 4N_0\hat{b}_0^+\hat{b}_0 + N_0e^{-2i\theta}\hat{b}_0\hat{b}_0 \right] \\ & - \frac{|V_0|}{V} N_0 \left[4 \sum_{q \neq 0} \hat{b}_q^+ \hat{b}_q + e^{2i\theta} \sum_{q \neq 0} \hat{b}_q^+ \hat{b}_{-q}^+ + e^{-2i\theta} \sum_{q \neq 0} \hat{b}_q \hat{b}_{-q} \right] \end{aligned}$$

The classical portion, with $\hat{K} = \hat{K}_{cl} + \hat{K}_q$, is

$$\hat{K}_{cl} = -\frac{|V_0|}{V} N_0^2 + [\varepsilon(0) - \mu] N_0$$

and so

$$\frac{\partial \hat{K}_{cl}}{\partial N_0} = 0$$

gives

$$2 \frac{|V_0|}{V} N_0 = \varepsilon(0) - \mu.$$

Collecting terms linear in \hat{b}, \hat{b}^+ , then, I have:

$$\begin{aligned} \hat{K}(linear) &= \sqrt{N_0} e^{-i\theta} \left[\varepsilon(0) - \mu - 2N_0 \frac{|V_0|}{V} \right] \hat{b}_0 + \sqrt{N_0} e^{i\theta} \left[\varepsilon(0) - \mu - 2N_0 \frac{|V_0|}{V} \right] \hat{b}_0^+ \\ &\rightarrow \sqrt{N_0} e^{-i\theta} [\varepsilon(0)] \hat{b}_0 + \sqrt{N_0} e^{i\theta} [\varepsilon(0)] \hat{b}_0^+ \end{aligned}$$

And so taking $\varepsilon(0) = 0$ gives vanishing of the terms linear in \hat{b}, \hat{b}^+ .

Now I have:

$$\begin{aligned} \hat{K} &= [\varepsilon(0) - \mu] N_0 - \frac{|V_0|}{V} N_0^2 + \sum_k [\varepsilon(k) - \mu] \hat{b}_k^+ \hat{b}_k \\ &\quad - \frac{|V_0|}{V} N_0 \left[4 \sum_q \hat{b}_q^+ \hat{b}_q + e^{2i\theta} \sum_q \hat{b}_q^+ \hat{b}_{-q}^+ + e^{-2i\theta} \sum_q \hat{b}_q \hat{b}_{-q} \right] \end{aligned}$$

The next step is to select a transformation that eliminates the $\hat{b}^+ \hat{b}^+$ and $\hat{b} \hat{b}$ terms. I take this transformation to be of the form:

$$\hat{b}_k^+ e^{i\theta} = \hat{c}_k \cosh \phi_k + \hat{c}_{-k}^+ \sinh \phi_k, \text{ where } \phi_k \text{ is an even function of } k.$$

Further, I assume that $[\hat{c}_n^+, \hat{c}_m] = \delta_{n,m}$.

Under this ansatz, I have:

$$\begin{aligned}
\hat{K} &= [\varepsilon(0) - \mu]N_0 - \frac{|V_0|}{V} N_0^2 \\
&+ \sum_k [\varepsilon(k) - \mu] (\hat{c}_k \cosh \phi_k + \hat{c}_{-k}^+ \sinh \phi_k) (\hat{c}_k^+ \cosh \phi_k + \hat{c}_{-k} \sinh \phi_k) \\
&- \frac{|V_0|}{V} N_0 \left[\begin{aligned} &4 \sum_k (\hat{c}_k \cosh \phi_k + \hat{c}_{-k}^+ \sinh \phi_k) (\hat{c}_k^+ \cosh \phi_k + \hat{c}_{-k} \sinh \phi_k) \\ &+ \sum_k (\hat{c}_k \cosh \phi_k + \hat{c}_{-k}^+ \sinh \phi_k) (\hat{c}_{-k} \cosh \phi_k + \hat{c}_k^+ \sinh \phi_k) \\ &+ \sum_k (\hat{c}_k^+ \cosh \phi_k + \hat{c}_{-k} \sinh \phi_k) (\hat{c}_{-k}^+ \cosh \phi_k + \hat{c}_k \sinh \phi_k) \end{aligned} \right]
\end{aligned}$$

Or simplifying,

$$\begin{aligned}
\hat{K} &= [\varepsilon(0) - \mu]N_0 - \frac{|V_0|}{V} N_0^2 \\
&+ \sum_k [\varepsilon(k) - \mu] (\hat{c}_k \hat{c}_k^+ \cosh^2 \phi_k + \hat{c}_{-k}^+ \hat{c}_k^+ \cosh \phi_k \sinh \phi_k + \hat{c}_k \hat{c}_{-k} \cosh \phi_k \sinh \phi_k + \hat{c}_{-k}^+ \hat{c}_{-k} \sinh^2 \phi_k) \\
&- \frac{|V_0|}{V} N_0 \left[\begin{aligned} &4 \sum_k (\hat{c}_k \hat{c}_k^+ \cosh^2 \phi_k + \hat{c}_{-k}^+ \hat{c}_k^+ \cosh \phi_k \sinh \phi_k + \hat{c}_k \hat{c}_{-k} \cosh \phi_k \sinh \phi_k + \hat{c}_{-k}^+ \hat{c}_{-k} \sinh^2 \phi_k) \\ &+ \sum_k (\hat{c}_k \hat{c}_{-k} \cosh^2 \phi_k + \hat{c}_{-k}^+ \hat{c}_{-k} \cosh \phi_k \sinh \phi_k + \hat{c}_k \hat{c}_k^+ \cosh \phi_k \sinh \phi_k + \hat{c}_{-k}^+ \hat{c}_k^+ \sinh^2 \phi_k) \\ &+ \sum_k (\hat{c}_k^+ \hat{c}_{-k}^+ \cosh^2 \phi_k + \hat{c}_k^+ \hat{c}_k \cosh \phi_k \sinh \phi_k + \hat{c}_{-k} \hat{c}_{-k}^+ \cosh \phi_k \sinh \phi_k + \hat{c}_{-k} \hat{c}_k \sinh^2 \phi_k) \end{aligned} \right]
\end{aligned}$$

Breaking the terms up into the desirable and undesirable ones, I have:

$$\begin{aligned}
\hat{K} &= [\varepsilon(0) - \mu]N_0 - \frac{|V_0|}{V} N_0^2 \\
&+ \sum_k \left[[\varepsilon(k) - \mu] (\cosh^2 \phi_k + \sinh^2 \phi_k) - \frac{|V_0|}{V} N_0 (4 \sinh^2 \phi_k + 4 \cosh^2 \phi_k + 4 \cosh \phi_k \sinh \phi_k) \right] \hat{c}_k^+ \hat{c}_k \\
&+ \sum_k \left[[\varepsilon(k) - \mu] \cosh \phi_k \sinh \phi_k - \frac{|V_0|}{V} N_0 (4 \cosh \phi_k \sinh \phi_k + \sinh^2 \phi_k + \cosh^2 \phi_k) \right] \hat{c}_{-k}^+ \hat{c}_k^+ \\
&+ \sum_k \left[[\varepsilon(k) - \mu] \cosh \phi_k \sinh \phi_k - \frac{|V_0|}{V} N_0 (4 \cosh \phi_k \sinh \phi_k + \sinh^2 \phi_k + \cosh^2 \phi_k) \right] \hat{c}_{-k} \hat{c}_k \\
&- \sum_k \left[[\varepsilon(k) - \mu] \cosh^2 \phi_k + 2 \frac{|V_0|}{V} N_0 \cosh \phi_k \sinh \phi_k \right]
\end{aligned}$$

Now with the understanding that the $\hat{c}^+ \hat{c}^+$ and $\hat{c} \hat{c}$ terms must vanish in order for my transformation to be successful, I expect to see two conditions:

$$[\varepsilon(k) - \mu] \cosh \phi_k \sinh \phi_k - \frac{|V_0|}{V} N_0 (4 \cosh \phi_k \sinh \phi_k + \sinh^2 \phi_k + \cosh^2 \phi_k) = 0$$

$$\cosh^2 \phi_k - \sinh^2 \phi_k = 1$$

Armed with these, I may now solve:

$$\frac{1}{2} [\varepsilon(k) - \mu] \sinh 2\phi_k - \frac{|V_0|}{V} N_0 (2 \sinh 2\phi_k + \cosh 2\phi_k) = 0$$

$$\left(\frac{1}{2} [\varepsilon(k) - \mu] - 2 \frac{|V_0|}{V} N_0 \right) \sinh 2\phi_k - \frac{|V_0|}{V} N_0 \cosh 2\phi_k = 0$$

Now certainly:

$$\sinh 2\phi_k \propto -\frac{|V_0|}{V} N_0$$

$$\cosh 2\phi_k \propto \frac{1}{2} [\varepsilon(k) - \mu] - 2 \frac{|V_0|}{V} N_0$$

Since $\cosh^2 \phi_k - \sinh^2 \phi_k = 1$, I determine that

$$\cosh 2\phi_k = \frac{\frac{1}{2} [\varepsilon(k) - \mu] - 2 \frac{|V_0|}{V} N_0}{\sqrt{\left(\frac{1}{2} [\varepsilon(k) - \mu] - 2 \frac{|V_0|}{V} N_0 \right)^2 - \left(-\frac{|V_0|}{V} N_0 \right)^2}}$$

$$\sinh 2\phi_k = \frac{-\frac{|V_0|}{V} N_0}{\sqrt{\left(\frac{1}{2} [\varepsilon(k) - \mu] - 2 \frac{|V_0|}{V} N_0 \right)^2 - \left(-\frac{|V_0|}{V} N_0 \right)^2}}$$

So that at last I have:

$$\hat{K} = [\varepsilon(0) - \mu]N_0 - \frac{|V_0|}{V} N_0^2 + \sum_k \left[[\varepsilon(k) - \mu] \cosh 2\phi_k - \frac{|V_0|}{V} N_0 (4 \cosh 2\phi_k + 2 \sinh 2\phi_k) \right] \hat{c}_k^+ \hat{c}_k$$

$$- \sum_k \left[[\varepsilon(k) - \mu] \cosh^2 \phi_k + 2 \frac{|V_0|}{V} N_0 \cosh \phi_k \sinh \phi_k \right]$$

where

$$\hbar\Omega(k) = [\varepsilon(k) - \mu] \cosh 2\phi_k - \frac{|V_0|}{V} N_0 (4 \cosh 2\phi_k + 2 \sinh 2\phi_k)$$

$$K_0 = - \sum_k \left[\frac{1}{2} [\varepsilon(k) - \mu] (\cosh 2\phi_k + 1) + \frac{|V_0|}{V} N_0 \sinh 2\phi_k \right] + [\varepsilon(0) - \mu]N_0 - \frac{|V_0|}{V} N_0^2$$

Next take $\varepsilon(k) = \frac{\hbar^2 k^2}{2m}$ for free particles.

Now consider the domain of $\hbar\Omega(k)$. With the Gross-Ginzburg-Pitaevskii condition that

$$2 \frac{|V_0|}{V} N_0 = \varepsilon(0) - \mu \text{ and taking } \varepsilon(0) = 0, \text{ I have:}$$

$$\hbar\Omega(k) = \varepsilon(k) \cosh 2\phi_k - 2 \frac{|V_0|}{V} N_0 (\cosh 2\phi_k + \sinh 2\phi_k)$$

Now with all substitutions intact, I have:

$$\hbar\Omega(k) = \left(\frac{\hbar^2 k^2}{2m} - 2 \frac{|V_0|}{V} N_0 \right) \frac{\frac{1}{2} \frac{\hbar^2 k^2}{2m} - \frac{|V_0|}{V} N_0}{\sqrt{\left(\frac{1}{2} \frac{\hbar^2 k^2}{2m} - \frac{|V_0|}{V} N_0 \right)^2 - \left(\frac{|V_0|}{V} N_0 \right)^2}} - 2 \frac{|V_0|}{V} N_0 \frac{-\frac{|V_0|}{V} N_0}{\sqrt{\left(\frac{1}{2} \frac{\hbar^2 k^2}{2m} - \frac{|V_0|}{V} N_0 \right)^2 - \left(\frac{|V_0|}{V} N_0 \right)^2}}$$

$$= 2 \frac{\left(\frac{1}{2} \frac{\hbar^2 k^2}{2m} - \frac{|V_0|}{V} N_0 \right)^2 - \left(\frac{|V_0|}{V} N_0 \right)^2}{\sqrt{\left(\frac{1}{2} \frac{\hbar^2 k^2}{2m} - \frac{|V_0|}{V} N_0 \right)^2 - \left(\frac{|V_0|}{V} N_0 \right)^2}} = 2 \sqrt{\left(\frac{1}{2} \frac{\hbar^2 k^2}{2m} - \frac{|V_0|}{V} N_0 \right)^2 - \left(\frac{|V_0|}{V} N_0 \right)^2}$$

This is clearly pure imaginary where

$$\left(\frac{1}{2} \frac{\hbar^2 k^2}{2m} - \frac{|V_0|}{V} N_0 \right)^2 - \left(\frac{|V_0|}{V} N \right)^2 < 0$$

$$\frac{1}{4} \left(\frac{\hbar^2 k^2}{2m} \right)^2 - \frac{\hbar^2 k^2}{2m} \frac{|V_0|}{V} N_0 < 0$$

$$\frac{1}{4} \frac{\hbar^2 k^2}{2m} < \frac{|V_0|}{V} N_0$$

b) What is k_{\max} ? What is the physical reason for this imaginary value and what do they mean?

The maximum pure-imaginary value is just below $\frac{1}{4} \frac{\hbar^2 k_{\max}^2}{2m} = \frac{|V_0|}{V} N_0$, after which the value of the \hat{K} operator is real. Again, I take $[\hat{c}_n^+, \hat{c}_m] = \delta_{n,m}$.

Certainly,

$$i\hbar \frac{d}{dt} \hat{c}_k = [\hat{H}, \hat{c}_k] = [\hat{K} + \mu \hat{N}, \hat{c}_k] = (\hbar\Omega(k) + \mu) \hat{c}_k \approx \hbar\Omega(k) \hat{c}_k.$$

Then, the solutions to this equation of motion are

$$\frac{d}{dt} \hat{c}_k = \frac{\hbar\Omega(k)}{i\hbar} \hat{c}_k \therefore \hat{c}_k \approx e^{\frac{\hbar\Omega(k)}{i\hbar} t}$$

Then, a mode in the range

$$\frac{1}{4} \frac{\hbar^2 k_{\max}^2}{2m} \leq \frac{|V_0|}{V} N_0$$

corresponding to

$$0 < k < k_{\max} = \frac{8m}{\hbar^2} \frac{|V_0|}{V} N_0$$

indicates a mode decaying in time, which is part of the condensate background. Above this background momentum, a mode will be free to propagate in the background.

2) Consider the same problem as 1 but now with $V(\vec{x} - \vec{y}) = \frac{e^2}{|\vec{x} - \vec{y}|}$, the

Coulomb potential. Namely, consider a weakly interacting Bose gas of charged particles (and assume a homogeneous neutralizing background like in the Jellium model).

- a) Obtain the energy eigenvalues $\hbar\Omega(k)$ in the Bogoliubov approximation. Show that now $\lim_{k \rightarrow 0} \Omega(k) = \Omega(0) \neq 0$. What is $\Omega(0)$? Compare to the result for plasma oscillations in the electron gas.

Taking everything as in problem 1, the key difference is that now:

$$\begin{aligned} & \iint d^3x d^3y \hat{\psi}^+(\bar{x}) \hat{\psi}^+(\bar{y}) V(\bar{x} - \bar{y}) \hat{\psi}(\bar{y}) \hat{\psi}(\bar{x}) \\ &= \int d^3x d^3y \frac{e^2}{|\bar{x} - \bar{y}|} \hat{\psi}^+(\bar{x}) \hat{\psi}^+(\bar{y}) \hat{\psi}(\bar{y}) \hat{\psi}(\bar{x}) \end{aligned}$$

The Fourier transform of the 3D Coulomb interaction leads to

$$V_{q \neq 0} = \frac{4\pi e^2}{q^2}$$

in the Jellium model, so that through the same procedure as in problem 1, I will be left with:

$$\iint d^3x d^3y \hat{\psi}^+(\bar{x}) \hat{\psi}^+(\bar{y}) V(\bar{x} - \bar{y}) \hat{\psi}(\bar{y}) \hat{\psi}(\bar{x}) \rightarrow -\frac{1}{2V} \sum_{q \neq 0, k, l} V_q \hat{a}_{l+q}^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_l.$$

There are no excitations corresponding to V_0 , since this interaction only occurs for $q = 0$. Now I may split up the interaction term a bit:

$$\begin{aligned} & -\frac{1}{2V} \sum_{q \neq 0, k, l} V_q \hat{a}_{l+q}^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_l = \\ & -\frac{1}{2V} \sum_{q \neq 0} V_q \hat{a}_q^+ \hat{a}_{-q}^+ \hat{a}_0 \hat{a}_0 - \frac{1}{2V} \sum_{q \neq 0, k \neq 0, k \neq q} V_q \hat{a}_q^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_0 - \frac{1}{2V} \sum_{q \neq 0, k \neq 0, l \neq 0, k \neq q, l \neq -q} V_q \hat{a}_{l+q}^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_l \\ & -\frac{1}{2V} \sum_{q \neq 0, l \neq 0, l \neq -q} V_q \hat{a}_{l+q}^+ \hat{a}_{-q}^+ \hat{a}_0 \hat{a}_l - \frac{1}{2V} \sum_{q \neq 0} V_q \hat{a}_q^+ \hat{a}_0^+ \hat{a}_q \hat{a}_0 - \frac{1}{2V} \sum_{q \neq 0, l \neq 0, l \neq -q} V_q \hat{a}_{l+q}^+ \hat{a}_0^+ \hat{a}_q \hat{a}_l \\ & -\frac{1}{2V} \sum_{q \neq 0} V_q \hat{a}_0^+ \hat{a}_{-q}^+ \hat{a}_0 \hat{a}_{-q} - \frac{1}{2V} \sum_{q \neq 0, k \neq 0, k \neq q} V_q \hat{a}_0^+ \hat{a}_{k-q}^+ \hat{a}_k \hat{a}_{-q} - \frac{1}{2V} \sum_{q \neq 0} V_q \hat{a}_0^+ \hat{a}_0^+ \hat{a}_q \hat{a}_{-q} \end{aligned}$$

Under the substitution $\hat{a}_0 = \sqrt{N_0} e^{i\theta} + \hat{b}_0$ $\hat{a}_{k \neq 0} = \hat{b}_{k \neq 0}$, then, the only terms surviving in the Bogoliubov transformation correspond to those with at least two zero-indexed operators \hat{a}_0, \hat{a}_0^+ . Then, what remains is:

$$\hat{H}_I = -\frac{N_0}{V} \sum_{q \neq 0} V_q \hat{b}_q^+ \hat{b}_q - \frac{N_0}{2V} e^{2i\theta} \sum_{q \neq 0} V_q \hat{b}_q^+ \hat{b}_{-q}^+ - \frac{N_0}{2V} e^{-2i\theta} \sum_{q \neq 0} V_q \hat{b}_q \hat{b}_{-q} + \theta(\sqrt{N}) + \theta(1)$$

Now writing $\hat{K} = \hat{H} - \mu\hat{N}$, I see that the Gross-Ginzburg-Pitaevskii condition is unnecessary to eliminate terms linear in \hat{b} since none appear, and so I correspondingly take $\mu = 0$.

So now I have:

$$\hat{K} = \hat{H} = \sum_k \left[\varepsilon(k) - \delta_{k \neq 0} \frac{N_0}{V} V_k \right] \hat{b}_k^+ \hat{b}_k - \frac{N_0}{2V} e^{2i\theta} \sum_{k \neq 0} V_k \hat{b}_k^+ \hat{b}_{-k}^+ - \frac{N_0}{2V} e^{-2i\theta} \sum_{k \neq 0} V_k \hat{b}_k \hat{b}_{-k}$$

I would like a transformation of the form

$$\hat{b}_k^+ e^{i\theta} = \hat{c}_k u(k) + \hat{c}_{-k}^+ v(k), \text{ where } u \text{ and } v \text{ are even functions of } k.$$

Writing this in terms of these operators, then:

$$\begin{aligned} \hat{K} = \hat{H} &= \sum_k \left[\varepsilon(k) - \delta_{k \neq 0} \frac{N_0}{V} V_k \right] (\hat{c}_k u(k) + \hat{c}_{-k}^+ v(k)) (\hat{c}_k^+ u^*(k) + \hat{c}_{-k} v^*(k)) \\ &- \frac{N_0}{2V} \sum_{k \neq 0} V_k (\hat{c}_k u(k) + \hat{c}_{-k}^+ v(k)) (\hat{c}_{-k} u(k) + \hat{c}_k^+ v(k)) - \frac{N_0}{2V} \sum_{k \neq 0} V_k (\hat{c}_k^+ u^*(k) + \hat{c}_{-k} v^*(k)) (\hat{c}_{-k}^+ u^*(k) + \hat{c}_k v^*(k)) \\ &= \sum_k \left[\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right) (|v_k|^2 + |u_k|^2) - \delta_{k \neq 0} \frac{N_0 V_k}{V} (u(k)v(k) + u^*(k)v^*(k)) \right] \hat{c}_k^+ \hat{c}_k \\ &+ \sum_k \left[\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right) u^*(k)v(k) - \delta_{k \neq 0} \frac{N_0 V_k}{2V} (v(k)^2 + u^*(k)^2) \right] \hat{c}_{-k}^+ \hat{c}_k^+ \\ &+ \sum_k \left[\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right) u(k)v^*(k) - \delta_{k \neq 0} \frac{N_0 V_k}{2V} (v^*(k)^2 + u(k)^2) \right] \hat{c}_{-k} \hat{c}_k \\ &+ \sum_k \text{const.ants} \end{aligned}$$

As shown in a previous homework assignment, for a Bose gas in the Bogoliubov approximation, I expect $|u(k)|^2 - |v(k)|^2 = 1$ in order to ensure canonical commutation relations.

In order for the undesired $\hat{c}_{-k}^+ \hat{c}_k^+$ and $\hat{c}_{-k} \hat{c}_k$ terms to vanish, then, I need:

$$\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right) u^*(k)v(k) - \delta_{k \neq 0} \frac{N_0 V_k}{2V} (v(k)^2 + u^*(k)^2) = 0.$$

This implies that:

$$u^*(k)v(k) \propto \delta_{k \neq 0} \frac{N_0 V_k}{2V}$$

$$v(k)^2 + u^*(k)^2 \propto \varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V}$$

These relations can be satisfied by entirely real u and v , for:

$$u^*(k)v(k) = u(k)v(k) \propto \delta_{k \neq 0} \frac{N_0 V_k}{2V}$$

$$v(k)^2 + u^*(k)^2 = v(k)^2 + u(k)^2 \propto \varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V}$$

In order to satisfy the canonical commutation relations, I must normalize these so that $|u(k)|^2 - |v(k)|^2 = 1$, leaving:

$$u(k) = \kappa(k) \frac{\sqrt{2} \delta_{k \neq 0} \frac{N_0 V_k}{2V}}{\sqrt{\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right) - \sqrt{\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right)^2 - 4 \left(\delta_{k \neq 0} \frac{N_0 V_k}{2V} \right)^2}}}$$

$$v(k) = \kappa(k) \frac{1}{\sqrt{2}} \frac{\sqrt{\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right) - \sqrt{\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right)^2 - 4 \left(\delta_{k \neq 0} \frac{N_0 V_k}{2V} \right)^2}}}{\sqrt{\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right)^2 - 4 \left(\delta_{k \neq 0} \frac{N_0 V_k}{2V} \right)^2}}}$$

$$\kappa(k) = \frac{1}{\left(\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right)^2 - 4 \left(\delta_{k \neq 0} \frac{N_0 V_k}{2V} \right)^2 \right)^{\frac{1}{4}}}$$

Under this substitution,

$$\begin{aligned} \hbar\Omega(k) &= \left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right) (v_k^2 + u_k^2) - \delta_{k \neq 0} \frac{2N_0 V_k}{V} u(k)v(k) \\ &= \frac{\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right)^2 - 4 \left(\delta_{k \neq 0} \frac{N_0 V_k}{2V} \right)^2}{\left(\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right)^2 - 4 \left(\delta_{k \neq 0} \frac{N_0 V_k}{2V} \right)^2 \right)^{\frac{1}{4}}} \\ &= \sqrt{\left(\varepsilon(k) - \delta_{k \neq 0} \frac{N_0 V_k}{V} \right)^2 - \left(\delta_{k \neq 0} \frac{N_0 V_k}{V} \right)^2} \end{aligned}$$

Taking the limit as $k \rightarrow 0$, I get:

$$\lim_{k \rightarrow 0} \hbar\Omega(k) = \lim_{k \rightarrow 0} \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \frac{N_0 V_k}{V} \right)^2 - \left(\frac{N_0 V_k}{V} \right)^2} = \lim_{k \rightarrow 0} \sqrt{\left(\frac{\hbar^2 k^2}{2m} \right)^2 - \frac{4\pi e^2 \hbar^2 N_0}{mV}} = i\hbar \sqrt{\frac{4\pi e^2 N_0}{mV}}$$

So, I see that this agrees exactly with the plasma frequency for a free electron gas:

In a free electron gas, $\omega_{pl} = \sqrt{\frac{4\pi N_0 e^2}{mV}}$ [http://en.wikipedia.org/wiki/Plasma_frequency]

b) Give the behavior of the Bogoliubov coefficients for small k . This is the case of long range forces that violates Goldstone's Theorem.

Using the forms derived above, I get, allowing Mathematica to do the tedious but

$$u(k \approx 0) = (-1)^{\frac{5}{4}} \sqrt{\frac{1}{5} \left(\frac{\pi e^2 m N_0}{V} \right)^{\frac{1}{4}}} \frac{1}{k} + \dots$$

$$v(k \approx 0) = (-1)^{\frac{1}{4}} \sqrt{\frac{1}{5} \left(\frac{\pi e^2 m N_0}{V} \right)^{\frac{1}{4}}} \frac{1}{k} + \dots$$

Note that a phase has appeared in these, whereas I assumed that they were real in order to solve above. The solution above is certainly accurate for large k , and there is no reason to believe that a solution accurate for that region wouldn't extend naturally into the small k region.

So I notice two that there is a symmetry between the lowest order term contribution of my two coefficients, namely that they differ only by a constant factor of -1 , thus these are candidates for Goldstone's Theorem which states that any theory with an exact symmetry other than that of the vacuum must contain a massless particle.

3) Pairing theory of Superconductivity: The BCS Hamiltonian in the mean field approximation is

$$\hat{K} = \hat{H} - \mu \hat{N} = \sum_k \varepsilon(k) (\hat{a}_{k\uparrow}^+ \hat{a}_{k\uparrow} + \hat{a}_{k\downarrow}^+ \hat{a}_{k\downarrow}) + \sum_k [\Delta \hat{a}_{k\uparrow}^+ \hat{a}_{-k\downarrow}^+ + \Delta^* \hat{a}_{-k\downarrow} \hat{a}_{k\uparrow}]$$

where $\hat{a}_{k\uparrow,\downarrow}^+$ are creation operators of an electron of spin up or down and momentum \vec{k} and

$$\Delta = -\frac{g}{V} \sum_{k'} \langle GS | \hat{a}_{-k\downarrow} \hat{a}_{k\uparrow} | GS \rangle.$$

$|GS\rangle$ is the ground state of \hat{K} and $\sum_{k'}$ is a sum over states with

$$0 \leq \varepsilon(k') \leq \hbar \omega_m \quad \text{and} \quad \varepsilon(k) = \varepsilon(k) - \mu = \frac{\hbar^2 k^2}{2m} - \mu.$$

a) Diagonalize \hat{K} by a Bogoliubov transformation: introduce new operators

$$\hat{A}_k = u_k \hat{a}_{k\uparrow} - v_k \hat{a}_{-k\downarrow}^+$$

$$\hat{B}_k^+ = v_k \hat{a}_{k\uparrow} + u_k \hat{a}_{-k\downarrow}^+$$

and their respective Hermitian conjugates with u_k, v_k real functions and even in k . Show that this transformation is canonical, namely that \hat{A}, \hat{B} obey the

usual anti-commutation relations if $u_k^2 + v_k^2 = 1$. It is convenient to write

$$u_k = \left[\frac{1}{2}(1 + \alpha_k) \right]^{\frac{1}{2}} \quad v_k = - \left[\frac{1}{2}(1 - \alpha_k) \right]^{\frac{1}{2}}.$$

Invert the Bogoliubov transformation and write \hat{K} in terms of $\hat{A}, \hat{A}^+, \hat{B}, \hat{B}^+$.

Choose α_k so that \hat{K} becomes $\hat{K} = \sum_k E(k) [\hat{A}_k^+ \hat{A}_k + \hat{B}_k^+ \hat{B}_k] + K_0$. Namely,

choose α_k to make terms such as $\hat{A}\hat{B}$ vanish. Find $E(k)$.

$$\begin{aligned} \{\hat{A}_k, \hat{B}_k^+\} &= \{u_k \hat{a}_{k\uparrow} - v_k \hat{a}_{-k\downarrow}^+, v_k \hat{a}_{k\uparrow} + u_k \hat{a}_{-k\downarrow}^+\} = 0 \\ \{\hat{A}_k, \hat{B}_k\} &= \{u_k \hat{a}_{k\uparrow} - v_k \hat{a}_{-k\downarrow}^+, v_k \hat{a}_{k\uparrow} + u_k \hat{a}_{-k\downarrow}^+\} = u_k v_k \{\hat{a}_{k\uparrow}, \hat{a}_{k\uparrow}^+\} - v_k u_k \{\hat{a}_{-k\downarrow}^+, \hat{a}_{-k\downarrow}^+\} = 0 \\ \{\hat{A}_k, \hat{A}_k^+\} &= \{u_k \hat{a}_{k\uparrow} - v_k \hat{a}_{-k\downarrow}^+, u_k \hat{a}_{k\uparrow}^+ - v_k \hat{a}_{-k\downarrow}^+\} = u_k^2 \{\hat{a}_{k\uparrow}, \hat{a}_{k\uparrow}^+\} + v_k^2 \{\hat{a}_{-k\downarrow}^+, \hat{a}_{-k\downarrow}^+\} = u_k^2 + v_k^2 = 1 \\ \{\hat{B}_k, \hat{B}_k^+\} &= \{v_k \hat{a}_{k\uparrow} + u_k \hat{a}_{-k\downarrow}^+, v_k \hat{a}_{k\uparrow} + u_k \hat{a}_{-k\downarrow}^+\} = v_k^2 \{\hat{a}_{k\uparrow}, \hat{a}_{k\uparrow}^+\} + u_k^2 \{\hat{a}_{-k\downarrow}^+, \hat{a}_{-k\downarrow}^+\} = v_k^2 + u_k^2 = 1 \end{aligned}$$

Thus I see that this is a canonical transformation.

Certainly, then,

$$\begin{aligned} \hat{a}_{k\uparrow} &= u_k \hat{A}_k + v_k \hat{B}_k^+ \\ \hat{a}_{-k\downarrow}^+ &= -v_k \hat{A}_k + u_k \hat{B}_k^+ \end{aligned}$$

Now,

$$\begin{aligned} \hat{K} &= \hat{H} - \mu \hat{N} = \sum_k \varepsilon(k) (\hat{a}_{k\uparrow}^+ \hat{a}_{k\uparrow} + \hat{a}_{k\downarrow}^+ \hat{a}_{k\downarrow}) + \sum_k [\Delta \hat{a}_{k\uparrow}^+ \hat{a}_{-k\downarrow}^+ + \Delta^* \hat{a}_{-k\downarrow} \hat{a}_{k\uparrow}] \\ &= \sum_k \varepsilon(k) \left((u_k \hat{A}_k^+ + v_k \hat{B}_k) (u_k \hat{A}_k + v_k \hat{B}_k^+) + (-v_k \hat{A}_k + u_k \hat{B}_k^+) (-v_k \hat{A}_k^+ + u_k \hat{B}_k) \right) \\ &\quad + \sum_k \left[\Delta (u_k \hat{A}_k^+ + v_k \hat{B}_k) (-v_k \hat{A}_k + u_k \hat{B}_k^+) + \Delta^* (-v_k \hat{A}_k^+ + u_k \hat{B}_k) (u_k \hat{A}_k + v_k \hat{B}_k^+) \right] \end{aligned}$$

Separating this into component terms, then, I have:

$$\begin{aligned} \hat{K} &= \sum_k \left[\varepsilon(k) (u_k^2 - v_k^2) - (\Delta + \Delta^*) u_k v_k \right] \hat{A}_k^+ \hat{A}_k \\ &\quad + \sum_k \left[\varepsilon(k) (u_k^2 - v_k^2) - (\Delta + \Delta^*) u_k v_k \right] \hat{B}_k^+ \hat{B}_k \\ &\quad + \sum_k \left[\varepsilon(k) (2u_k v_k) + (\Delta u_k^2 - \Delta^* v_k^2) \right] \hat{A}_k^+ \hat{B}_k^+ \\ &\quad + \sum_k \left[\varepsilon(k) (-2u_k v_k) - (\Delta^* u_k^2 - \Delta v_k^2) \right] \hat{A}_k \hat{B}_k \\ &\quad + \text{const.ants} \end{aligned}$$

Now in order to ensure vanishing of $\hat{A}\hat{B}$ terms, I need:

$$\varepsilon(k)(2u_k v_k) + (\Delta u_k^2 - \Delta^* v_k^2) = \varepsilon(k)(-2u_k v_k) - (\Delta^* u_k^2 - \Delta v_k^2) = 0$$

Next I write:

$$\varepsilon(k)(1 + \alpha_k)^{\frac{1}{2}}(1 - \alpha_k)^{\frac{1}{2}} + \frac{\Delta}{2}(1 + \alpha_k) - \frac{\Delta^*}{2}(1 - \alpha_k) = 0$$

$$\varepsilon(k)\left((1 + \alpha_k)^{\frac{1}{2}}(1 - \alpha_k)^{\frac{1}{2}}\right) + \frac{\Delta + \Delta^*}{2}\alpha_k = 0$$

$$\left(\frac{\Delta + \Delta^*}{2}\right)^2 \alpha_k^2 = \varepsilon(k)^2(1 - \alpha_k^2)$$

$$\alpha_k = \frac{\varepsilon(k)}{\sqrt{\left(\frac{\Delta + \Delta^*}{2}\right)^2 + \varepsilon(k)^2}}$$

Now I see that

$$E(k) = \varepsilon(k)(u_k^2 - v_k^2) - (\Delta + \Delta^*)u_k v_k$$

with

$$u_k = \left[\frac{1}{2}(1 + \alpha_k)\right]^{\frac{1}{2}} \quad v_k = -\left[\frac{1}{2}(1 - \alpha_k)\right]^{\frac{1}{2}} \quad \alpha_k = \frac{\varepsilon(k)}{\sqrt{\left(\frac{\Delta + \Delta^*}{2}\right)^2 + \varepsilon(k)^2}}$$

- b) Obtain a self-consistent equation for Δ by evaluating $\langle GS | \hat{a}_{-k\downarrow} \hat{a}_{k\uparrow} | GS \rangle$ and solve this equation by replacing $\frac{1}{V} \sum_{k'} \rightarrow \int_0^{\hbar\omega_m} N(\varepsilon) d\varepsilon$ with $N(\varepsilon)$ is the density of states so that $N(\varepsilon) d\varepsilon = \frac{d^3k}{(2\pi)^3}$ and assume $N(\varepsilon) \approx N(0)$.**

Then,

$$\begin{aligned}
\Delta &= -\frac{g}{V} \sum_{k'} \langle GS | \hat{a}_{-k\downarrow} \hat{a}_{k\uparrow} | GS \rangle \rightarrow gN(0) \int_0^{\hbar\omega_m} \frac{d^3k}{(2\pi)^3} \langle GS | \hat{a}_{-k\downarrow} \hat{a}_{k\uparrow} | GS \rangle \\
&= gN(0) \int_0^{\hbar\omega_m} \frac{d^3k}{(2\pi)^3} \langle GS | \left(-v_k \hat{A}_k^+ + u_k \hat{B}_k \right) \left(u_k \hat{A}_k + v_k \hat{B}_k^+ \right) | GS \rangle \\
&= gN(0) \int_0^{\hbar\omega_m} \frac{d^3k}{(2\pi)^3} \langle GS | \left(-v_k u_k \hat{A}_k^+ \hat{A}_k - v_k^2 \hat{A}_k^+ \hat{B}_k^+ + u_k^2 \hat{B}_k \hat{A}_k - v_k u_k \hat{B}_k^+ \hat{B}_k \right) | GS \rangle \\
&\text{(const.ant dropped)} \\
&\text{only } \hat{B}_k^+ \hat{B}_k \text{ term survives} \\
&= -gN(0) \int_0^{\hbar\omega_m} \frac{d^3k}{(2\pi)^3} v_k u_k
\end{aligned}$$

Where:

$$\begin{aligned}
v_k u_k &= -\left[\frac{1}{2}(1-\alpha_k) \right]^{\frac{1}{2}} \left[\frac{1}{2}(1+\alpha_k) \right]^{\frac{1}{2}} = -\frac{1}{2} [(1-\alpha_k)(1+\alpha_k)]^{\frac{1}{2}} = -\frac{1}{2} [1-\alpha_k^2]^{\frac{1}{2}} \\
&= -\frac{1}{2} \left[1 - \frac{\varepsilon(k)^2}{\left(\frac{\Delta + \Delta^*}{2} \right)^2 + \varepsilon(k)^2} \right]^{\frac{1}{2}} = -\frac{1}{2} \left[\frac{(\text{Re}(\Delta))^2}{(\text{Re}(\Delta))^2 + \varepsilon(k)^2} \right]^{\frac{1}{2}} = -\frac{1}{2} \frac{\text{Re}(\Delta)}{\sqrt{(\text{Re}(\Delta))^2 + \varepsilon(k)^2}}
\end{aligned}$$

So

$$\Delta = -gN(0) \int_0^{\hbar\omega_m} \frac{d^3k}{(2\pi)^3} v_k u_k = \frac{gN(0)}{2} \int_0^{\hbar\omega_m} \frac{d^3k}{(2\pi)^3} \frac{\text{Re}(\Delta)}{\sqrt{(\text{Re}(\Delta))^2 + \varepsilon(k)^2}}$$

This self-consistency condition ensures that in this regime, Δ must be real.

c) Obtain the distribution function $\langle GS | \hat{a}_{k\uparrow}^+ \hat{a}_{k\uparrow} | GS \rangle$

$$\begin{aligned}
& \langle GS | \hat{a}_{k\uparrow}^+ \hat{a}_{k\uparrow} | GS \rangle \\
&= \langle GS | \left(u_k \hat{A}_k^+ + v_k \hat{B}_k \right) \left(u_k \hat{A}_k + v_k \hat{B}_k^+ \right) | GS \rangle \\
&= \langle GS | \left(u_k^2 \hat{A}_k^+ \hat{A}_k + u_k v_k \hat{A}_k^+ \hat{B}_k^+ + u_k v_k \hat{B}_k \hat{A}_k + v_k^2 \hat{B}_k \hat{B}_k^+ \right) | GS \rangle \\
& \text{(const. ant dropped)}
\end{aligned}$$

only $\hat{B}_k^+ \hat{B}_k$ term survives

$$= -v_k^2$$

$$= -\frac{1}{2}(1 - \alpha_k) = -\frac{1}{2} \left(1 - \frac{\varepsilon(k)}{\sqrt{\left(\frac{\Delta + \Delta^*}{2}\right)^2 + \varepsilon(k)^2}} \right)$$

d) Show that $E(0) = \Delta \equiv \text{gap}$ and evaluate the resulting integral in part (b) to give the gap Δ as a function of $gN(0)$ for $gN(0) \ll 1$.

Under the substitution $k \rightarrow 0$, and combining results from parts a and b,

$$\begin{aligned}
E(0) &= \varepsilon(0)(u_0^2 - v_0^2) - (\Delta + \Delta^*)u_0v_0 \\
&= -2 \operatorname{Re}[\Delta]u_0v_0 = -2 \operatorname{Re}[\Delta] \left(-\frac{1}{2} \frac{\operatorname{Re}[\Delta]}{\sqrt{\operatorname{Re}[\Delta]^2}} \right) = \operatorname{Re}[\Delta] = \Delta
\end{aligned}$$

Finishing, I take the integral:

$$\Delta \equiv \operatorname{Re}[\Delta]$$

$$\Delta = \frac{gN(0)}{2} \int_0^{\hbar\omega_m} d\varepsilon \frac{\Delta}{\sqrt{\Delta^2 + \varepsilon^2}} = \frac{gN(0)}{2} \Delta \arcsin h \left(\frac{\hbar\omega_m}{\Delta} \right)$$

$$1 = \frac{gN(0)}{2} \operatorname{arc. sinh} \left(\frac{\hbar\omega_m}{\Delta} \right)$$

$$\Delta = \frac{\hbar\omega_m}{\sinh \left(\frac{2}{gN(0)} \right)}$$

$$gN(0) \ll 1: \Delta \rightarrow 2\hbar\omega_m e^{-\frac{2}{gN(0)}}$$

4) Polaritons in Semiconductors: In semiconductors, particles and holes bind together via their mutual Coulomb attraction into particle-hole bound states called excitons. These are bosonic excitations described by creation and

annihilation operators \hat{B}_q where \bar{q} is the center of mass momentum of the bound state, but the energy ε_0 is independent of the \bar{q} - momentum. The free exciton Hamiltonian is $\hat{H}_B = \sum_q \varepsilon_0 \hat{B}_q^+ \hat{B}_q$; \hat{B}, \hat{B}^+ obey the canonical commutation relations for Bosons. These particle-hole bound states interact with photons via their dipole moment. The total exciton-photon Hamiltonian is $\hat{H} = \sum_q \varepsilon_0 \hat{B}_q^+ \hat{B}_q + \sum_q \hbar \omega_q \hat{b}_q^+ \hat{b}_q - ig \sum_q (\hat{B}_q^+ \hat{b}_q - \hat{b}_q^+ \hat{B}_q)$ with \hat{b}_q the photon operators (only one polarization is considered for simplicity) and g is a real coupling constant with $\frac{g}{\varepsilon_0} \ll 1$, and $\omega_q = cq$.

a) Introduce polariton operators:

$$\hat{\rho}_{1q} = u_q \hat{B}_q - v_q \hat{b}_q$$

$$\hat{\rho}_{2q} = v_q^* \hat{B}_q + u_q^* \hat{b}_q$$

What is the condition for u_q, v_q so that the polariton operators obey canonical commutation relations?

In order to obey canonical commutation relations, I must have $[\hat{\rho}_n^+, \hat{\rho}_m] = \delta_{n,m}$. Then, assuming that the operators \hat{B}_q and \hat{b}_q themselves obey canonical commutation relations,

$$[\hat{\rho}_{1q}^+, \hat{\rho}_{1q}] = [u_q^* \hat{B}_q^+ - v_q^* \hat{b}_q^+, u_q \hat{B}_q - v_q \hat{b}_q] = |u_q|^2 + |v_q|^2$$

$$[\hat{\rho}_{2q}^+, \hat{\rho}_{2q}] = [v_q \hat{B}_q^+ + u_q \hat{b}_q^+, v_q^* \hat{B}_q + u_q^* \hat{b}_q] = |u_q|^2 + |v_q|^2$$

so that I must have $|u_q|^2 + |v_q|^2 = 1$.

b) Write the Hamiltonian in terms of the polariton operators and choose u_q, v_q so that the terms of the form $\hat{\rho}_1^+ \hat{\rho}_2, \hat{\rho}_2^+ \hat{\rho}_1$ vanish. Show that with the proper choice of u_q, v_q ,

$$\hat{H} = \sum_q \Omega_1(q) \hat{\rho}_{1q}^+ \hat{\rho}_{1q} + \sum_q \Omega_2(q) \hat{\rho}_{2q}^+ \hat{\rho}_{2q}$$

Plot $\Omega_{1,2}$ vs. q and show that as $q \rightarrow \infty$ one is photon-like and the other is exciton-like.

Inverting these relationships, then:

$$\begin{aligned} \hat{\rho}_{1q} &= u_q \hat{B}_q - v_q \hat{b}_q & \hat{\rho}_{1q}^+ &= u_q^* \hat{B}_q^+ - v_q^* \hat{b}_q^+ \\ \hat{\rho}_{2q} &= v_q^* \hat{B}_q + u_q^* \hat{b}_q & \hat{\rho}_{2q}^+ &= v_q \hat{B}_q^+ + u_q \hat{b}_q^+ \end{aligned}$$

gives

$$\begin{aligned}\hat{B}_q &= v_q \hat{\rho}_{2q} + u_q^* \hat{\rho}_{1q} & \hat{B}_q^+ &= v_q^* \hat{\rho}_{2q}^+ + u_q \hat{\rho}_{1q}^+ \\ \hat{b}_q &= u_q \hat{\rho}_{2q} - v_q^* \hat{\rho}_{1q} & \hat{b}_q^+ &= u_q^* \hat{\rho}_{2q}^+ - v_q \hat{\rho}_{1q}^+\end{aligned}$$

Now substituting this into

$$\hat{H} = \sum_q \varepsilon_0 \hat{B}_q^+ \hat{B}_q + \sum_q \hbar \omega_q \hat{b}_q^+ \hat{b}_q - ig \sum_q \left(\hat{B}_q^+ \hat{b}_q - \hat{b}_q^+ \hat{B}_q \right),$$

I have:

$$\begin{aligned}\hat{H} &= \sum_q \varepsilon_0 \left(v_q^* \hat{\rho}_{2q}^+ + u_q \hat{\rho}_{1q}^+ \right) \left(v_q \hat{\rho}_{2q} + u_q^* \hat{\rho}_{1q} \right) \\ &+ \sum_q \hbar \omega_q \left(u_q^* \hat{\rho}_{2q}^+ - v_q \hat{\rho}_{1q}^+ \right) \left(u_q \hat{\rho}_{2q} - v_q^* \hat{\rho}_{1q} \right) \\ &- ig \sum_q \left(\left(v_q^* \hat{\rho}_{2q}^+ + u_q \hat{\rho}_{1q}^+ \right) \left(u_q \hat{\rho}_{2q} - v_q^* \hat{\rho}_{1q} \right) - \left(u_q^* \hat{\rho}_{2q}^+ - v_q \hat{\rho}_{1q}^+ \right) \left(v_q \hat{\rho}_{2q} + u_q^* \hat{\rho}_{1q} \right) \right)\end{aligned}$$

Separating these into the wanted and unwanted terms, I have:

$$\begin{aligned}\hat{H} &= \sum_q \left[\varepsilon_0 |v_q|^2 + \hbar \omega_q |u_q|^2 - ig v_q^* u_q + ig u_q^* v_q \right] \hat{\rho}_{2q}^+ \hat{\rho}_{2q} \\ &+ \sum_q \left[\varepsilon_0 |u_q|^2 + \hbar \omega_q |v_q|^2 + ig u_q v_q^* - ig v_q u_q^* \right] \hat{\rho}_{1q}^+ \hat{\rho}_{1q} \\ &+ \sum_q \left[\varepsilon_0 u_q v_q - \hbar \omega_q u_q v_q - ig u_q^2 - ig v_q^2 \right] \hat{\rho}_{1q}^+ \hat{\rho}_{2q} \\ &+ \sum_q \left[\varepsilon_0 u_q^* v_q^* - \hbar \omega_q u_q^* v_q^* + ig v_q^{*2} + ig u_q^{*2} \right] \hat{\rho}_{2q}^+ \hat{\rho}_{1q}\end{aligned}$$

Now I see that the condition

$$\left(\varepsilon_0 - \hbar \omega_q \right) u_q v_q - ig \left(u_q^2 + v_q^2 \right) = 0$$

leads to vanishing of the $\hat{\rho}_1^+ \hat{\rho}_2, \hat{\rho}_2^+ \hat{\rho}_1$ terms.

From this, it is clear that

$$u_q^2 + v_q^2 \propto \left(\varepsilon_0 - \hbar \omega_q \right)$$

$$u_q v_q \propto ig$$

Then, if

$$u_q = a_q + ib_q$$

$$v_q = c_q + id_q$$

then

$$a_q^2 - b_q^2 + c_q^2 - d_q^2 = \varepsilon_0 - \hbar\omega_q$$

$$a_q b_q + c_q d_q = 0$$

$$a_q c_q - b_q d_q = 0$$

$$b_q c_q + a_q d_q = g$$

with a normalization condition

$$a_q^2 + b_q^2 + c_q^2 + d_q^2 = 1$$

Solving this system of equations in Mathematica gives, pre-normalization:

$$u_q = \kappa(q) \frac{\sqrt{2}ig}{\sqrt{(\varepsilon_0 - \hbar\omega_q) + \sqrt{(\varepsilon_0 - \hbar\omega_q)^2 + 4g^2}}} \quad v_q = \kappa(q) \frac{1}{\sqrt{2}} \sqrt{(\varepsilon_0 - \hbar\omega_q) + \sqrt{(\varepsilon_0 - \hbar\omega_q)^2 + 4g^2}}$$

Normalizing, I find that

$$\kappa(q) = \frac{1}{\sqrt{|u_q|^2 + |v_q|^2}} = \frac{1}{\left((\varepsilon_0 - \hbar\omega_q)^2 + 4g^2\right)^{\frac{1}{4}}}$$

So that

$$u_q = \frac{1}{\left((\varepsilon_0 - \hbar\omega_q)^2 + 4g^2\right)^{\frac{1}{4}}} \frac{\sqrt{2}ig}{\sqrt{(\varepsilon_0 - \hbar\omega_q) + \sqrt{(\varepsilon_0 - \hbar\omega_q)^2 + 4g^2}}} \quad v_q = \frac{\sqrt{(\varepsilon_0 - \hbar\omega_q) + \sqrt{(\varepsilon_0 - \hbar\omega_q)^2 + 4g^2}}}{\sqrt{2}\left((\varepsilon_0 - \hbar\omega_q)^2 + 4g^2\right)^{\frac{1}{4}}}$$

At last,

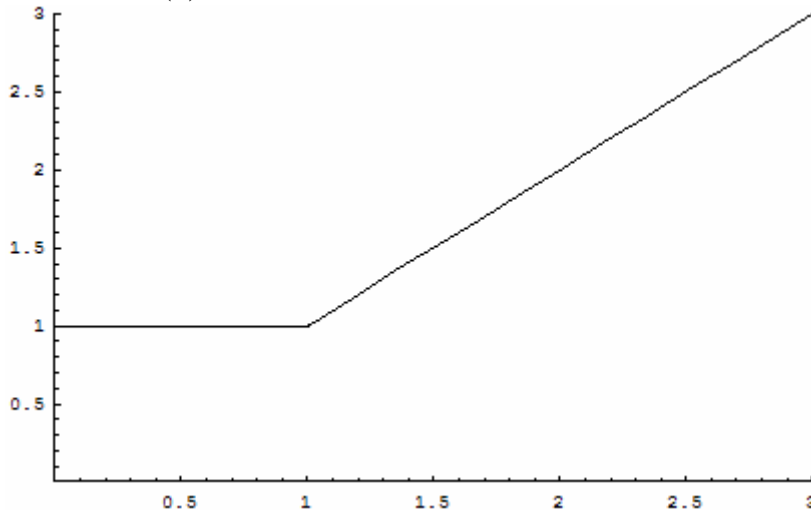
$$\Omega_2(q) = \varepsilon_0 |v_q|^2 + \hbar\omega_q |u_q|^2 - igv_q^* u_q + igu_q^* v_q$$

$$= \frac{1}{2} \left(\varepsilon_0 + \hbar\omega_q + \sqrt{4g^2 + (\varepsilon_0 - \hbar\omega_q)^2} \right)$$

$$\Omega_1(q) = \varepsilon_0 |u_q|^2 + \hbar\omega_q |v_q|^2 + igu_q v_q^* - igv_q u_q^*$$

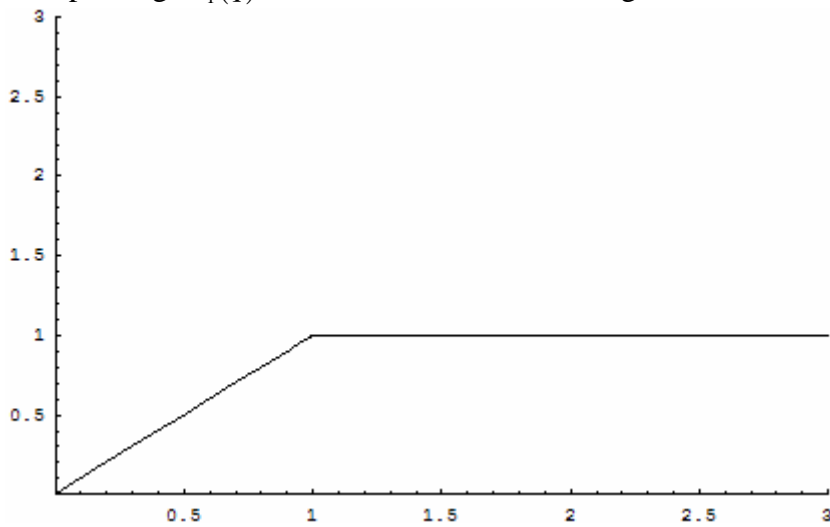
$$= \frac{1}{2} \left(\varepsilon_0 + \hbar\omega_q - \sqrt{4g^2 + (\varepsilon_0 - \hbar\omega_q)^2} \right)$$

Plotting $\Omega_2(q)$ with some token values gives:



This is clearly photon-like, since its energy is increasing linearly with $q \rightarrow \infty$.

And plotting $\Omega_1(q)$ with the same token values gives:



This is clearly exciton-like, since its energy is not varying with $q \rightarrow \infty$.

c) Give the $q \rightarrow \infty$ and $q \rightarrow 0$ limits for u_q, v_q .

$$\lim_{q \rightarrow 0} u_q = \lim_{q \rightarrow 0} \frac{1}{\left((\varepsilon_0 - \hbar\omega_q)^2 + 4g^2 \right)^{\frac{1}{4}}} \frac{\sqrt{2ig}}{\sqrt{(\varepsilon_0 - \hbar\omega_q) + \sqrt{(\varepsilon_0 - \hbar\omega_q)^2 + 4g^2}}} = \frac{1}{(\varepsilon_0^2 + 4g^2)^{\frac{1}{4}}} \frac{\sqrt{2ig}}{\sqrt{\varepsilon_0 + \sqrt{\varepsilon_0^2 + 4g^2}}}$$

$$\lim_{q \rightarrow 0} v_q = \lim_{q \rightarrow 0} \frac{\sqrt{(\varepsilon_0 - \hbar\omega_q) + \sqrt{(\varepsilon_0 - \hbar\omega_q)^2 + 4g^2}}}{\sqrt{2} \left((\varepsilon_0 - \hbar\omega_q)^2 + 4g^2 \right)^{\frac{1}{4}}} = \frac{\sqrt{\varepsilon_0 + \sqrt{\varepsilon_0^2 + 4g^2}}}{\sqrt{2} (\varepsilon_0^2 + 4g^2)^{\frac{1}{4}}}$$

$$\lim_{q \rightarrow \infty} u_q = \lim_{q \rightarrow \infty} \frac{1}{\left((\varepsilon_0 - \hbar\omega_q)^2 + 4g^2 \right)^{\frac{1}{4}}} \frac{\sqrt{2ig}}{\sqrt{(\varepsilon_0 - \hbar\omega_q) + \sqrt{(\varepsilon_0 - \hbar\omega_q)^2 + 4g^2}}} \approx \frac{1}{(\hbar\omega_q)^{\frac{1}{2}}} \frac{\sqrt{2ig}}{(\hbar\omega_q)^{\frac{1}{2}} \sqrt{2}g} \rightarrow i$$

$$\lim_{q \rightarrow \infty} v_q = \lim_{q \rightarrow \infty} \frac{\sqrt{(\varepsilon_0 - \hbar\omega_q) + \sqrt{(\varepsilon_0 - \hbar\omega_q)^2 + 4g^2}}}{\sqrt{2} \left((\varepsilon_0 - \hbar\omega_q)^2 + 4g^2 \right)^{\frac{1}{4}}} \approx \frac{(\hbar\omega_q)^{-\frac{1}{2}} \sqrt{2}g}{\sqrt{2} (\hbar\omega_q)^{\frac{1}{2}}} \rightarrow \frac{g}{\hbar\omega_q} \rightarrow 0$$