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Quantum Mechanics 2 Homework 3

2) (a) Consider the quantum history for $t_1 < t_2 < t_3$

$$Y = [\phi_1] \circ ([\phi_1] + [\phi_2]) \circ [\phi_5]$$

for a harmonic oscillator with Hilbert space H, where $[\phi_n]$ **projects onto the ray with energy** $\left(n + \frac{1}{2}\right)\hbar\omega$**. Find the smallest (fewest number of elements) or coarsest product sample space which contains Y as one of its elements.**

$$\{[\phi_1], I - [\phi_1]\} \circ \{[\phi_1] + [\phi_2], I - [\phi_1] - [\phi_2]\} \circ \{[\phi_5], I - [\phi_5]\}$$

b) **For the purposes of this exercise, let us say that any subspace of H with projector of the form** $\sum_n [\phi_n]$**, for n in an arbitrary collection, is an "energy subspace". Find a history Y' with non-trivial projectors (something other than I) at** t_1, t_2, t_3 **which commutes with Y, employs at least one projector onto a subspace which is not an energy subspace, and for which** $Z = Y'Y = YY'$ **is not zero and is different from both Y and Y'.**

$$Y' = [\phi_1] \circ (|\phi_1\rangle + |\phi_2\rangle)(\langle\phi_1| + \langle\phi_2|) \circ [\phi_5]$$

Each element commutes with the corresponding element in Y, and at least one is different than Y which makes $Z = Y'Y = YY'$ nonzero, and certainly the t_2 element is not an energy subspace.

c) **Find the smallest or coarsest sample space of histories such that the corresponding Boolean algebra contains both Y from (a) and Y' from (b). (Note that this need not be a product sample space.)**

$Y' = [\phi_1] \circ (|\phi_1\rangle + |\phi_2\rangle)(\langle\phi_1| + \langle\phi_2|) \circ [\phi_5]$ contains Y as a subset.

3) Consider a spin-half particle in zero magnetic field so that $T(t, t') = I$**, and let the initial state at** t_0 **be** $|\psi_0\rangle = |z^+\rangle$**. Consider a family of histories**

$$[z^+] \circ \{[\omega^+], [\omega^-]\} \circ \{[x^+], [x^-]\}$$

at times $t_0 < t_1 < t_2$**. For w = x, w = y, and w = z, determine whether the family is consistent, and if it is consistent, find the four conditional probabilities** $\Pr([w^\pm]_1, [x^\pm]_2 \mid [z^+]_0)$ **where the subscripts indicate the times of the events.**

(i) **w = x**

$$[x^\pm] = \frac{1}{\sqrt{2}}(|z^+\rangle \pm |z^-\rangle)(\langle z^+| \pm \langle z^-|)$$

$$\frac{1}{4}\left(|z^+\rangle \pm |z^-\rangle\right)\left(\langle z^+| \pm \langle z^-|\right)T\left(|z^+\rangle \pm |z^-\rangle\right)\left(\langle z^+| \pm \langle z^-|\right)T\left[z^+\right]z^+\rangle$$

$$++: |z^+\rangle \to |z^+\rangle + |z^-\rangle \to \frac{1}{\sqrt{2}}|z^+\rangle + \frac{1}{\sqrt{2}}|z^-\rangle : \frac{1}{2} = \Pr\left(x^+{}_1, x^+{}_2 \mid z^+{}_0\right)$$

$$+-: |z^+\rangle \to |z^+\rangle + |z^-\rangle \to |z^+\rangle - |z^-\rangle - |z^+\rangle + |z^-\rangle = 0 = \Pr\left(x^+{}_1, x^-{}_2 \mid z^+{}_0\right)$$

$$-+: |z^+\rangle \to |z^+\rangle - |z^-\rangle \to |z^+\rangle + |z^-\rangle - |z^+\rangle - |z^-\rangle = 0 = \Pr\left(x^-{}_1, x^+{}_2 \mid z^+{}_0\right)$$

$$--: |z^+\rangle \to |z^+\rangle - |z^-\rangle \to \frac{1}{\sqrt{2}}|z^+\rangle - \frac{1}{\sqrt{2}}|z^-\rangle : \frac{1}{2} = \Pr\left(x^-{}_1, x^-{}_2 \mid z^+{}_0\right)$$

these are orthogonal and therefore compatible.

(ii) w = y

$$\left[y^\pm\right] = \frac{1}{\sqrt{2}}\left(|z^+\rangle \pm i|z^-\rangle\right)\left(\langle z^+| \mp i\langle z^-|\right)$$

$$\frac{1}{2}\left(|z^+\rangle \pm |z^-\rangle\right)\left(\langle z^+| \pm \langle z^-|\right)T\left(|z^+\rangle \pm i|z^-\rangle\right)\left(\langle z^+| \mp i\langle z^-|\right)T\left[z^+\right]z^+\rangle$$

$$++: |z^+\rangle \to |z^+\rangle + i|z^-\rangle \to \frac{1}{2}|z^+\rangle + \frac{1}{2}|z^-\rangle + \frac{1}{2}i|z^+\rangle + \frac{1}{2}i|z^-\rangle$$

$$+-: |z^+\rangle \to |z^+\rangle + i|z^-\rangle \to \frac{1}{2}|z^+\rangle - \frac{1}{2}|z^-\rangle + \frac{1}{2}i|z^+\rangle + \frac{1}{2}i|z^-\rangle$$

Not Orthogonal

Not Consistent

(iii) w = z

$$\left[z^\pm\right] = |z^\pm\rangle\langle z^\pm|$$

$$\frac{1}{2}\left(|z^+\rangle \pm |z^-\rangle\right)\left(\langle z^+| \pm \langle z^-|\right)T|z^\pm\rangle\langle z^\pm|T\left[z^+\right]z^+\rangle$$

$$++: |z^+\rangle \to |z^+\rangle \to \frac{1}{\sqrt{2}}|z^+\rangle + \frac{1}{\sqrt{2}}|z^-\rangle : \frac{1}{2} = \Pr\left(z^+{}_1, x^+{}_2 \mid z^+{}_0\right)$$

$$+-: |z^+\rangle \to |z^+\rangle \to \frac{1}{\sqrt{2}}|z^+\rangle - \frac{1}{\sqrt{2}}|z^-\rangle : \frac{1}{2} = \Pr\left(z^+{}_1, x^-{}_2 \mid z^+{}_0\right)$$

$$-+: |z^+\rangle \to |z^+\rangle \to 0 = \Pr\left(z^-{}_1, x^+{}_2 \mid z^+{}_0\right)$$

$$--: |z^+\rangle \to |z^+\rangle - 0 = \Pr\left(z^-{}_1, x^-{}_2 \mid z^+{}_0\right)$$

b) Next suppose that the time development operator is

$$T(t_2, t_1) = T(t_1, t_0) = U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \langle z^+|U|z^+\rangle & \langle z^+|i|z^-\rangle \\ \langle z^-|U|z^+\rangle & \langle z^-|1|z^-\rangle \end{pmatrix}$$

Find two choices for w, giving rise to two different decompositions of the identity $\left\{\left[w^+\right], \left[w^-\right]\right\}$, such that the family $\left[z^+\right] \circ \left\{\left[\omega^+\right], \left[\omega^-\right]\right\} \circ \left\{\left[x^+\right], \left[x^-\right]\right\}$ is consistent.

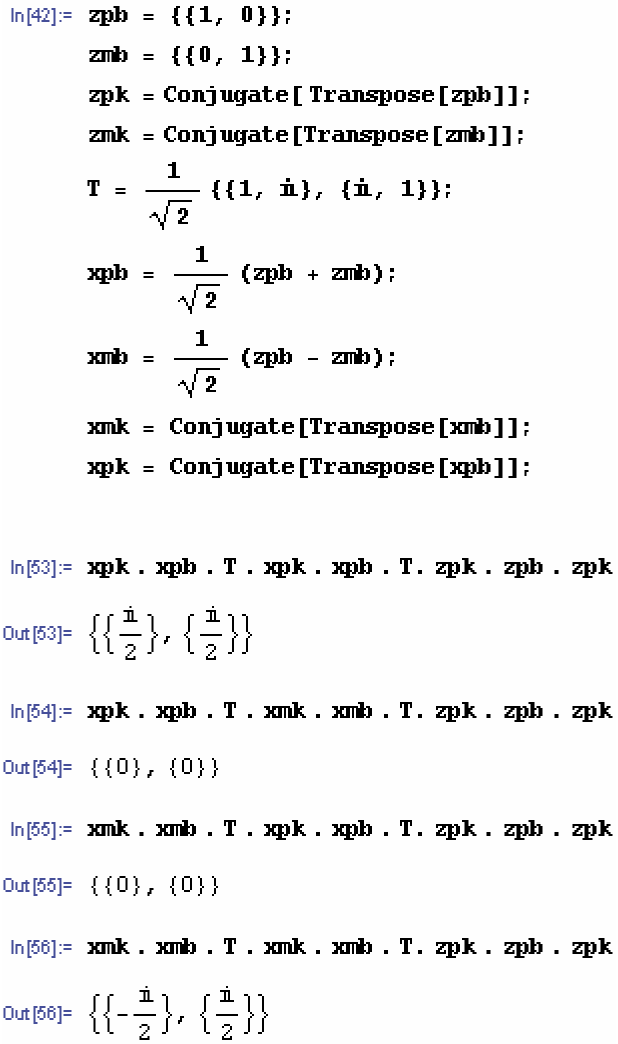
Now the time operator takes $\begin{matrix} \left|z^+\right\rangle \rightarrow \frac{1}{\sqrt{2}}\left|z^+\right\rangle + \frac{1}{\sqrt{2}}i\left|z^-\right\rangle \\ \left|z^-\right\rangle \rightarrow \frac{1}{\sqrt{2}}i\left|z^+\right\rangle + \frac{1}{\sqrt{2}}\left|z^-\right\rangle \end{matrix}$, and does the same with the kets changed into the same bras.

Perhaps some simplification will make the choices clear:

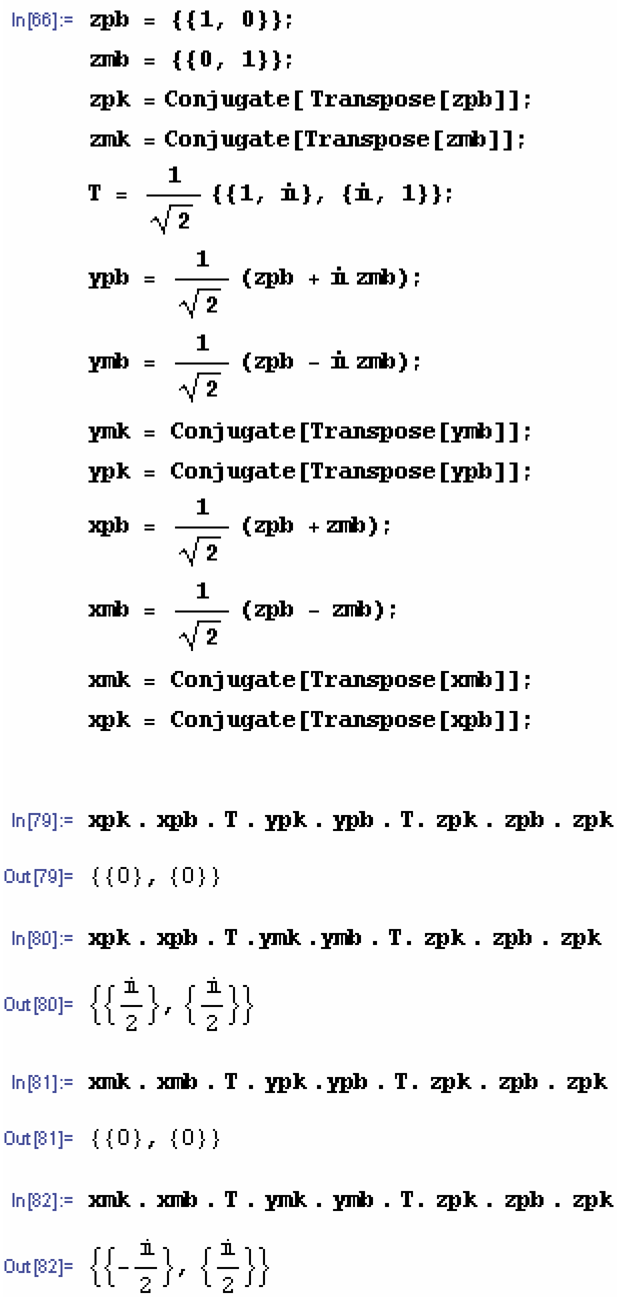
$$\frac{1}{\sqrt{2}}\left(\left|z^+\right\rangle \pm \left|z^-\right\rangle\right)\left(\left\langle z^+\right| \pm \left\langle z^-\right|\right)T\left\{\left[\omega^+\right],\left[\omega^-\right]\right\}T\left[z^+\right]$$

$$= \frac{1}{2\sqrt{2}}\left(\left|z^+\right\rangle \pm \left|z^-\right\rangle\right)\left((1+i)\left\langle z^-\right| + (1\pm i)\left\langle z^+\right|\right)\left\{\left[\omega^+\right],\left[\omega^-\right]\right\}\left(\left|z^+\right\rangle + i\left|z^-\right\rangle\right)\left\langle z^+\right|$$

Looking at the forms of these, w = x and w = y look like very plausible choices as each will collapse one of the kets. Evaluating this looks like it's going to be very time-consuming, so I'll do it in Mathematica. Clearly x works, as the results are orthogonal:



```
In[42]:= zpb = {{1, 0}};
         zmb = {{0, 1}};
         zpk = Conjugate[ Transpose[zpb]];
         zmk = Conjugate[Transpose[zmb]];
         T = 1/√2 {{1, ⅈ}, {ⅈ, 1}};
         xpb = 1/√2 (zpb + zmb);
         xmb = 1/√2 (zpb - zmb);
         xmk = Conjugate[Transpose[xmb]];
         xpk = Conjugate[Transpose[xpb]];

In[53]:= xpk . xpb . T . xpk . xpb . T. zpk . zpb . zpk

Out[53]= {{ⅈ/2}, {ⅈ/2}}

In[54]:= xpk . xpb . T . xmk . xmb . T. zpk . zpb . zpk

Out[54]= {{0}, {0}}

In[55]:= xmk . xmb . T . xpk . xpb . T. zpk . zpb . zpk

Out[55]= {{0}, {0}}

In[56]:= xmk . xmb . T . xmk . xmb . T. zpk . zpb . zpk

Out[56]= {{-ⅈ/2}, {ⅈ/2}}
```



```
In[66]:= zpb = {{1, 0}};
         zmb = {{0, 1}};
         zpk = Conjugate[Transpose[zpb]];
         zmk = Conjugate[Transpose[zmb]];
         T = 1/√2 {{1, ⅈ}, {ⅈ, 1}};
         ypb = 1/√2 (zpb + ⅈ zmb);
         ymb = 1/√2 (zpb - ⅈ zmb);
         ymk = Conjugate[Transpose[ymb]];
         ypk = Conjugate[Transpose[ypb]];
         xpb = 1/√2 (zpb + zmb);
         xmb = 1/√2 (zpb - zmb);
         xmk = Conjugate[Transpose[xmb]];
         xpk = Conjugate[Transpose[xpb]];

In[79]:= xpk . xpb . T . ypk . ypb . T. zpk . zpb . zpk

Out[79]= {{0}, {0}}

In[80]:= xpk . xpb . T .ymk .ymb . T. zpk . zpb . zpk

Out[80]= {{ⅈ/2}, {ⅈ/2}}

In[81]:= xmk . xmb . T . ypk .ypb . T. zpk . zpb . zpk

Out[81]= {{0}, {0}}

In[82]:= xmk . xmb . T . ymk . ymb . T. zpk . zpb . zpk

Out[82]= {{-ⅈ/2}, {ⅈ/2}}
```

and so clearly w = y also works.

4) **The consistency of a family with an initial state $|\omega_0\rangle$ at t_0 and orthonormal bases $\left\{|\phi_j^{\alpha_j}\rangle\right\}$, where the subscript indicates the time, at times $t_0 < t_1 < ... < t_f$ can be checked in the following way. Construct a graph consisting of a single node in the left-most column corresponding to $|\omega_0\rangle$, a set of nodes in the next column corresponding to $\left\{|\phi_1^{\alpha_1}\rangle\right\}$ for which $\langle\phi_1^{\alpha_1}|T(t_1,t_0)|\psi_0\rangle$ is not zero, connected to the $|\omega_0\rangle$ node with lines. In the next column place $\left\{|\phi_2^{\alpha_2}\rangle\right\}$ nodes, connect to the previous nodes when $\langle\phi_2^{\alpha_2}|T(t_2,t_1)|\phi_1^{\alpha_1}\rangle$ is non-zero, et**

cetera. Note that only nodes that are actually connected to a line on the previous level are included.

Show that the family is consistent if and only if the graph constructed in this way has no closed loops.

First, note that all kets multiplied by a constant non-zero factor will behave the same under time transformation, and recover the factor at the end. This means that any node reached at a level from a previous one will recover some portion of the nodes connected to it at the final step.

Now, certainly, any two paths that share any portion of any final node will not be orthogonal under the product. Also note that if two paths converge anywhere, then since the time operator is blind to the history before this point, they must follow the same path henceforth from the node at which convergence occurs and so share a portion of a final node. Therefore, if any paths converge then the histories must not be consistent.

Now suppose that no paths crossed. Along any branch of the tree, then, I get a component of each endpoint at the leaves of the tree. Now every path that has any probability of occurring at all is represented using some constants times each of these leaves at the final time. Since these kets are all orthogonal and no branch that terminates shares any of them, then, all results must be orthogonal and therefore the family must be consistent.

6) Use the result of Problem 4 to show that in Problem 3 the two choices of w that you found are unique: any other possibility would lead to an inconsistent family. Then show that there is a different choice for $T(t_2, t_1) = T(t_1, t_0)$ (keep the two equal to each other) for which there is only one possibility for the intermediate time decomposition which will yield a consistent family.

Empirically, I see that the space for the final projector is only two dimensional, and so based on that two of my probabilities must be zero at the end (otherwise they cannot all be orthogonal in two dimensions). This leaves me with one of two options: one is that either $\left\langle x^+ \middle| T^2 \middle| w^- \right\rangle = 0$ or $\left\langle x^+ \middle| T^2 \middle| w^+ \right\rangle = 0$ and either $\left\langle x^- \middle| T^2 \middle| w^+ \right\rangle = 0$ or $\left\langle x^- \middle| T^2 \middle| w^- \right\rangle = 0$. The other option would be for at least one of $\left\langle w^\pm \middle| T^2 \middle| z^+ \right\rangle = 0$ so that now it is plain to see that this implies the existence of one of two orthogonality conditions on each of $T^2 \left| w^\pm \right\rangle$ (since these projectors sum to 1, the choice of constraints for x is arbitrary—the w obtained will still be essentially the same). Now this choice of orthogonality conditions shows that there must be two unique solutions, which I have found. (w = y takes the former route, w = x takes the latter route.)

Now suppose I want to enforce the existence of only one unique solution. In order to do this, I need to either make one constraint impossible or to collapse these two options to be, in fact, one. The former is not realistic: a unitary transformation can only take a vector to another vector (never to more than one), so making one or another constraint "impossible" can't be done with unitary matrices. However, making the two constraints look ultimately alike also does not seem possible: Suppose I found a matrix such that $T^2\left|x^\pm\right\rangle = \left|z^\pm\right\rangle$, for $\left\langle z^\pm\right|T^2\left|x^\pm\right\rangle = \delta_{i,j}$ (where T is in the z basis). This would make the constraints identical and therefore make only one choice of w exist. One possibility is $T^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

7) **Consider a two-track toy model, where the particle can be either on track a or track b, and T = S with $S|m,z\rangle = |(m+1),z\rangle$ for z = a or b. For each of the four cases given below construct a consistent family which includes the initial state $|\psi_0\rangle$ at t = 0 followed by the later events as specified, if it is possible to do so. If it is not possible, explain why. In cases where a consistent family exists, find appropriate decompositions of the identity at t =1 and t = 2, indicate which histories form the support of the family, and find the associated probabilities, conditional on the initial state $|\psi_0\rangle$**

(i) $|\psi_0\rangle = \dfrac{|0a\rangle + 2|0b\rangle}{\sqrt{5}} \quad (t=1): |\lambda\rangle = |1a\rangle \quad (t=2): |\mu\rangle = |2b\rangle$

Choose as my projectors:

$(t=1): [1a], I-[1a] \quad (t=2): [2b], I-[2b].$

$$[2b]T[1a]T\frac{|0a\rangle + 2|0b\rangle}{\sqrt{5}} = [2b]T\frac{|1a\rangle}{\sqrt{5}} = 0 : \Pr(1a \cap 2b \mid \psi_0) = 0$$

$$[2b]T(I-[1a])T\frac{|0a\rangle + 2|0b\rangle}{\sqrt{5}} = [2b]T\left(\frac{2|1b\rangle}{\sqrt{5}}\right) = [2b]T\left(\frac{2|2b\rangle}{\sqrt{5}}\right) : \Pr(\overline{1a} \cap 2b \mid \psi_0) = \frac{4}{5}$$

$$(I-[2b])T[1a]T\frac{|0a\rangle + 2|0b\rangle}{\sqrt{5}} = (I-[2b])T\frac{|1a\rangle}{\sqrt{5}} = \frac{|1a\rangle}{\sqrt{5}} : \Pr(1a \cap \overline{2b} \mid \psi_0) = \frac{1}{5}$$

$$(I-[2b])T(I-[1a])T\frac{|0a\rangle + 2|0b\rangle}{\sqrt{5}} = (I-[2b])T\left(\frac{2|1b\rangle}{\sqrt{5}}\right) = 0 : \Pr(\overline{1a} \cap \overline{2b} \mid \psi_0) = 0$$

While a framework exists where either of these events are possible, both happening in the same history is not. If 1a occurs, nothing further on the b-track may.

(ii) $|\psi_0\rangle = |0a\rangle \quad (t=1): |\lambda\rangle = \dfrac{|1a\rangle + |1b\rangle}{\sqrt{2}} \quad (t=2): |\mu\rangle = |2b\rangle$

Observe that $\langle\psi_0| \circ I \circ [2b]$ is zero, but $\langle\psi_0| \circ [\lambda] \circ [2b]$ is nonzero. Therefore, no consistent family can exist with these events.

(iii) $|\psi_0\rangle = |0a\rangle \quad (t=1): |\lambda\rangle = \dfrac{|1a\rangle + |1b\rangle}{\sqrt{2}} \quad (t=2): |\mu\rangle = \dfrac{|2a\rangle - |2b\rangle}{\sqrt{2}}$

$$(t=1):\frac{1}{\sqrt{2}}\left(|1a\rangle+|1b\rangle\right)\left(\langle 1a|+\langle 1b|\right), I-\frac{1}{\sqrt{2}}\left(|1a\rangle+|1b\rangle\right)\left(\langle 1a|+\langle 1b|\right)$$

$$(t=2):\frac{1}{\sqrt{2}}\left(|2a\rangle-|2b\rangle\right), I-\frac{1}{\sqrt{2}}\left(|2a\rangle-|2b\rangle\right)$$

$$\frac{1}{2}\left(|2a\rangle-|2b\rangle\right)\left(\langle 2a|-\langle 2b|\right)T\left(|1a\rangle+|1b\rangle\right)\left(\langle 1a|+\langle 1b|\right)T|0a\rangle=$$

$$\frac{1}{2}\left(|2a\rangle-|2b\rangle\right)\left(\langle 2a|-\langle 2b|\right)T\left(|1a\rangle+|1b\rangle\right)=\frac{1}{2}\left(\left(|2a\rangle-|2b\rangle\right)-\left(|2a\rangle-|2b\rangle\right)\right)=0=\Pr(\lambda\cap\mu\,|\,\psi_0)$$

$$\frac{1}{\sqrt{2}}\left[I-\frac{1}{\sqrt{2}}\left(|2a\rangle-|2b\rangle\right)\left(\langle 2a|-\langle 2b|\right)\right]T\left(|1a\rangle+|1b\rangle\right)\left(\langle 1a|+\langle 1b|\right)T|0a\rangle=$$

$$\frac{1}{\sqrt{2}}\left[I-\frac{1}{\sqrt{2}}\left(|2a\rangle-|2b\rangle\right)\left(\langle 2a|-\langle 2b|\right)\right]T\left(|1a\rangle+|1b\rangle\right)=$$

$$\frac{1}{\sqrt{2}}\left[\left(|2a\rangle+|2b\rangle\right)-\frac{1}{\sqrt{2}}\left(|2a\rangle-|2b\rangle\right)+\frac{1}{\sqrt{2}}\left(|2a\rangle-|2b\rangle\right)\right]=\frac{1}{\sqrt{2}}\left(|2a\rangle+|2b\rangle\right):\frac{1}{2}=\Pr(\lambda\cap\overline{\mu}\,|\,\psi_0)$$

$$\frac{1}{\sqrt{2}}\left(|2a\rangle-|2b\rangle\right)\left(\langle 2a|-\langle 2b|\right)T\left[I-\frac{1}{\sqrt{2}}\left(|1a\rangle+|1b\rangle\right)\left(\langle 1a|+\langle 1b|\right)\right]T|0a\rangle=$$

$$\frac{1}{\sqrt{2}}\left(|2a\rangle-|2b\rangle\right)\left(\langle 2a|-\langle 2b|\right)T\left[\frac{\sqrt{2}-1}{\sqrt{2}}|1a\rangle-\frac{1}{\sqrt{2}}|1b\rangle\right]=$$

$$\left[\frac{\sqrt{2}-1}{2}\left(|2a\rangle-|2b\rangle\right)-\frac{1}{2}\left(|2a\rangle-|2b\rangle\right)\right]=\frac{1}{\sqrt{2}}\left[|2a\rangle-|2b\rangle\right]:\frac{1}{2}=\Pr(\overline{\lambda}\cap\mu\,|\,\psi_0)$$

$$\left[I-\frac{1}{\sqrt{2}}\left(|2a\rangle-|2b\rangle\right)\left(\langle 2a|-\langle 2b|\right)\right]T\left[I-\frac{1}{\sqrt{2}}\left(|1a\rangle+|1b\rangle\right)\left(\langle 1a|+\langle 1b|\right)\right]T|0a\rangle=$$

$$\left[I-\frac{1}{\sqrt{2}}\left(|2a\rangle-|2b\rangle\right)\left(\langle 2a|-\langle 2b|\right)\right]T\left[\frac{\sqrt{2}-1}{\sqrt{2}}|1a\rangle-\frac{1}{\sqrt{2}}|1b\rangle\right]=$$

$$=\left[I-\frac{1}{\sqrt{2}}\left(|2a\rangle-|2b\rangle\right)\left(\langle 2a|-\langle 2b|\right)\right]\left[\frac{\sqrt{2}-1}{\sqrt{2}}|1a\rangle-\frac{1}{\sqrt{2}}|1b\rangle\right]=$$

$$\left[\frac{\sqrt{2}-1}{\sqrt{2}}|2a\rangle-\frac{1}{\sqrt{2}}|2b\rangle\right]-\left[\frac{\sqrt{2}-1}{2}|2a\rangle-\frac{\sqrt{2}-1}{2}|2b\rangle-\left[\frac{1}{2}|1a\rangle-\frac{1}{2}|1b\rangle\right]\right]=0=\Pr(\dot{\lambda}\cap\overline{\mu}\,|\,\psi_0)$$

(iv)

$$|\psi_0\rangle=\frac{|0a\rangle+2|0b\rangle}{\sqrt{5}}\quad(t=1):|\lambda\rangle=\frac{|1a\rangle+|2b\rangle}{\sqrt{2}}\quad(t=2):|\mu\rangle=\frac{-|2a\rangle+|1b\rangle}{\sqrt{2}}$$

$$(t=1): \frac{1}{\sqrt{2}}\left(|1a\rangle+|2b\rangle\right)\left(\langle 1a|+\langle 2b|\right), I-\frac{1}{\sqrt{2}}\left(|1a\rangle+|2b\rangle\right)\left(\langle 1a|+\langle 2b|\right)$$

$$(t=2): \frac{1}{\sqrt{2}}\left(-|2a\rangle+|1b\rangle\right)\left(-\langle 2a|+\langle 1b|\right), I-\frac{1}{\sqrt{2}}\left(-|2a\rangle+|1b\rangle\right)\left(-\langle 2a|+\langle 1b|\right)$$

$$\frac{1}{\sqrt{2}}\left(-|2a\rangle+|1b\rangle\right)\left(-\langle 2a|+\langle 1b|\right)T\frac{1}{\sqrt{2}}\left(|1a\rangle+|2b\rangle\right)\left(\langle 1a|+\langle 2b|\right)T\frac{|0a\rangle+2|0b\rangle}{\sqrt{5}}=$$

$$\frac{1}{\sqrt{5}}\frac{1}{2}\left(-|2a\rangle+|1b\rangle\right)\left(-\langle 2a|+\langle 1b|\right)T\left(|1a\rangle+|2b\rangle\right)=$$

$$\frac{1}{\sqrt{5}}\frac{1}{2}\left(-|2a\rangle+|1b\rangle\right)\left(-\langle 2a|+\langle 1b|\right)\left(|2a\rangle+|3b\rangle\right)=\frac{1}{\sqrt{5}}\frac{1}{2}\left(|2a\rangle-|1b\rangle\right): \Pr(\lambda\cap\mu\,|\,\psi_0)=\frac{2}{20}$$

$$(t=1): \frac{1}{\sqrt{2}}\left(|1a\rangle+|2b\rangle\right)\left(\langle 1a|+\langle 2b|\right), I-\frac{1}{\sqrt{2}}\left(|1a\rangle+|2b\rangle\right)\left(\langle 1a|+\langle 2b|\right)$$

$$(t=2): \frac{1}{\sqrt{2}}\left(-|2a\rangle+|1b\rangle\right)\left(-\langle 2a|+\langle 1b|\right), I-\frac{1}{\sqrt{2}}\left(-|2a\rangle+|1b\rangle\right)\left(-\langle 2a|+\langle 1b|\right)$$

$$\frac{1}{\sqrt{2}}\left(-|2a\rangle+|1b\rangle\right)\left(-\langle 2a|+\langle 1b|\right)T\left(I-\frac{1}{\sqrt{2}}\left(|1a\rangle+|2b\rangle\right)\left(\langle 1a|+\langle 2b|\right)\right)T\frac{|0a\rangle+2|0b\rangle}{\sqrt{5}}=$$

$$\frac{1}{\sqrt{2}\sqrt{5}}\left(-|2a\rangle+|1b\rangle\right)\left(-\langle 2a|+\langle 1b|\right)T\left(\frac{\sqrt{2}-1}{\sqrt{2}}|1a\rangle+2|1b\rangle-|2b\rangle\right)=$$

$$\frac{1}{\sqrt{2}\sqrt{5}}\left(-|2a\rangle+|1b\rangle\right)\left(-\langle 2a|+\langle 1b|\right)\left(\frac{\sqrt{2}-1}{\sqrt{2}}|2a\rangle+2|2b\rangle-|3b\rangle\right)=$$

$$\frac{1}{\sqrt{2}\sqrt{5}}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\left(-|2a\rangle+|1b\rangle\right)\right)= Not \quad Orthogonal$$

Not Consistent

8) **The following is a toy model of scattering in one dimension. The particle can be on either of two tracks, labeled a and b, with a position given by the integer m. The state $|ma\rangle$ is interpreted as meaning that the particle is at position x = m and moving to the right, while $|mb\rangle$ means x = m and the particle is moving to the left. The time development operator is T = S, where the shift S is:**

$S|ma\rangle=|(m+1)a\rangle, S|mb\rangle=|(m-1)b\rangle$ **with the following exceptions, which mimic the effect of a potential variable:**

$S|2a\rangle=\alpha|3a\rangle+\beta|2b\rangle, S|3b\rangle=\gamma|3a\rangle+\delta|2b\rangle$

a) **Use unitarity to put conditions on the amplitudes of reflection: What choice should be made for these four amplitudes so that the probability of a particle being transmitted through the barrier from the left to the right is equal to τ, a real number in the range 0 to 1?**

Using unitarity, there cannot possibly be more "probability" that the particle is present after transformation than before. Therefore, $\alpha^2 + \beta^2 = 1, \gamma^2 + \delta^2 = 1$ and certainly the probability of transmission corresponding to alpha must be $\alpha = \sqrt{\tau}, \beta = \sqrt{1-\tau}$

b) Show that the transmission coefficient for a particle moving from the right to the left is also equal to τ. Then show that the two transmission coefficients need not be the same if one replaces the quantum toy model with a classical hopping model associated with the same diagram, in which the hopping probabilities are equal to one in the direction of the arrows, except for the four arrows alpha, beta, gamma and delta, where these letters should be interpreted as real hopping probabilities satisfying $\alpha^2 + \beta^2 = 1, \gamma^2 + \delta^2 = 1$.

In order for the time transformation to be unitary, the columns (probability of leaving each cell) must all be orthonormal and also the rows (probability of entering each cell) must be orthonormal. Now since in this quantum case the unitarity is required, I see that the delta path leaving 3b and the beta path leaving 2a must sum to 1: thus, delta is constrained to be $\sqrt{\tau}$ **.**

In a classical hopping model following Markov processes, while all cells must certainly have a net probability of 1 of entering any other cell, there's no rule against unfairly depleting a given cell. Therefore, in this model delta and gamma could be whatever I like summing to one.

c) Suppose a two-state detector is added on the right side of the barrier, where it detects a particle as it hops from 3a to 4a, i.e., if the detector state is indicated by $|n\rangle$, then $T(3a, n) = |4a, 1-n\rangle$. Find the probability that the detector will have detected the particle at time t = 4, given the following initial state at t = 0:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|0a\rangle + |0b\rangle\right) \otimes |n = 0\rangle$$

at time 2, the two halves of my wave-function must certainly lie at 2a and -2b. Now observe that of particle a's density, τ goes straight through the barrier and $1-\tau$ is reflected, meaning that $1-\tau$ lies at 2b and τ at 3a. Now the τ part triggers the detector for a probability of $\frac{1}{2}\tau$ that the detector is triggered.

d) Given the initial state in c, and assuming that at time t = 4 the detector has detected the particle, what can you say about the particle's location, the value of m, for 0 < t < 4? Give an answer which as precise as possible, consistent with the laws of quantum mechanics and the simplifications inherent in a toy model.

Making explicit the time evolution, I have

$$\left|\psi_0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0a\right\rangle + \left|0b\right\rangle\right) \quad \left|\psi_1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|1a\right\rangle + \left|-1b\right\rangle\right) \quad \left|\psi_2\right\rangle = \frac{1}{\sqrt{2}}\left(\left|2a\right\rangle + \left|-2b\right\rangle\right)$$

$$\left|\psi_3\right\rangle = \frac{1}{\sqrt{2}}\left(\tau\left|3a\right\rangle + (1-\tau)\left|2b\right\rangle + \left|-3b\right\rangle\right) \quad \left|\psi_4\right\rangle = \frac{1}{\sqrt{2}}\left(\tau\left|4a\right\rangle + (1-\tau)\left|1b\right\rangle + \left|-4b\right\rangle\right)$$

So, if the detector has triggered, my history is certainly

$$\left|0a\right\rangle \to \left|1a\right\rangle \to \left|2a\right\rangle \to \left|3a\right\rangle \to \left|4a\right\rangle$$

which is the only history containing node 4a at all.