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Quantum Mechanics 2 Homework 2

2) The Hilbert space for the spin degrees of freedom of a deuterium atom is of the form $S \times N$, where S is the two-dimensional spin space of the electron and N the three-dimensional space of the deuteron. Let $|+\rangle$ and $|-\rangle$ be an orthonormal basis of S with

$$S_z|+\rangle = +\frac{1}{2}|+\rangle \quad S_z|-\rangle = -\frac{1}{2}|-\rangle$$

and $|+\rangle, |0\rangle, |-\rangle$ an orthonormal basis of N with

$$I_z|+\rangle = |+\rangle \quad I_z|0\rangle = |0\rangle \quad I_z|-\rangle = -|-\rangle$$

Define the three projectors

$$P = |-0\rangle\langle -0|$$

$$Q = |++\rangle\langle ++| + |+0\rangle\langle +0| + |+-\rangle\langle +-|$$

$$R = |+-\rangle\langle +-| + |--\rangle\langle --|$$

where $|-0\rangle$ stands for $|-\rangle \times |0\rangle$, etc.

a) What is the dimension of $S \times N$?

There are six orthogonal kets and therefore six dimensions:

$|++\rangle, |+0\rangle, |+-\rangle, |-+\rangle, |-0\rangle, |--\rangle$

b) Can any of these projectors be written in the form $A \times I$ or $I \times B$, where I denotes the identity operator? If so, find the corresponding A or B.

Operator Q can be written as such, with $A = |+\rangle\langle +|$ and I the identity decomposition of N, and R can be written as such, with $B = |-\rangle\langle -|$ and I the identity decomposition in terms of S.

c) Provide a physical interpretation of each of the projectors P, Q and R in terms of S_z or I_z or both.

P: The probability that the z-spin is down and the deuteron spin is neutral.
Q: The probability that the z-spin is up.
R: The probability that the deuteron spin is down.

d) Find the projectors $\tilde{Q}$ and $\tilde{R}$ corresponding to the negations of the projectors Q and R, and again give a physical interpretation.

$$\tilde{Q} = |-+\rangle\langle -+| + |-0\rangle\langle -0| + |--\rangle\langle --|$$

$$\tilde{R} = |++\rangle\langle ++| + |+0\rangle\langle +0| + |-+\rangle\langle -+| + |-0\rangle\langle -0|$$

In the Q case, I have the cases where the z-spin is negative. In the R case, I have the cases where the Deuteron spin is not down.

e) Find the projector K onto $S \times N$ that corresponds to the property $S_y = \frac{1}{2}$.

You may use the fact that for a spin-half particle the state $|+\rangle + i|-\rangle$ is an

eigenstate of S_y with eigenvalue $\frac{1}{2}$. Expressing K as a sum of dyads in a similar form to P, Q and R leads to a lengthy expression. You do not need to write it out as long as you make it plain that you know what you're doing.

The spin S_y is obviously independent of the Deuteron spin. Then I have

$$K = \frac{1}{2}\left(|+\rangle + i|-\rangle\right)\left[|+\rangle\langle+| + |0\rangle\langle 0| + |-\rangle\langle-|\right]\left(\langle+| - i\langle-|\right)$$

where the center items correspond to the identity decomposition of the Deuteron state.

f) With which of the projectors P, Q and R does K commute? Discuss this in terms of compatible and incompatible properties.

Without doing any calculation, I'd expect K to commute with R only. The y-spin does not commute with the z-spin, and therefore any projector that gives me information about the z-spin could not possibly commute with K, which give me information abut the y-spin. P and Q would both allow me to determine a property of the z-spin.

g) The smallest Boolean algebra containing both Q and R is generated by a particular decomposition of the identity on $S \times N$. What is this decomposition of the identity?

$$\tilde{Q} \cap \tilde{R} = |-0\rangle\langle -0| + |-+\rangle\langle -+| = |-\rangle\left[|0\rangle\langle 0| + |+\rangle\langle +|\right]\langle -|$$

$$Q \cap \tilde{R} = |++\rangle\langle ++| + |+0\rangle\langle +0| = |+\rangle\left[|0\rangle\langle 0| + |+\rangle\langle +|\right]\langle +|$$

$$\tilde{Q} \cap R = |--\rangle\langle --|$$

$$Q \cap R = |+-\rangle\langle +-|$$

$$I = \tilde{Q} \cap \tilde{R} + Q \cap \tilde{R} + \tilde{Q} \cap R + Q \cap R$$

So essentially, a minimalist could collapse the Deuteron state into just two projectors, $|-\rangle \quad and \quad |0\rangle + |+\rangle$ (note that I'd have to be careful if it wasn't given that I_z was orthonormal!)

3) The time-development operator for a three-state system is given by the matrix $T(t,t') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega(t-t') & i\sin\omega(t-t') \\ 0 & i\sin\omega(t-t') & \cos\omega(t-t') \end{pmatrix}$ **in a particular time-independent orthonormal basis, where omega is constant.**

a) Show that $T(t,t')$ is unitary, and use matrix multiplication to check that $T(t,t')T(t',t'') = T(t,t'')$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\omega(t-t') & i\sin\omega(t-t')\\ 0 & i\sin\omega(t-t') & \cos\omega(t-t')\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\omega(t-t') & i\sin\omega(t-t')\\ 0 & -i\sin\omega(t-t') & \cos\omega(t-t')\end{pmatrix}^{T*}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\omega(t-t') & i\sin\omega(t-t')\\ 0 & i\sin\omega(t-t') & \cos\omega(t-t')\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\omega(t-t') & -i\sin\omega(t-t')\\ 0 & -i\sin\omega(t-t') & \cos\omega(t-t')\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos^2\omega(t-t')+\sin^2\omega(t-t') & 0\\ 0 & 0 & \cos^2\omega(t-t')+\sin^2\omega(t-t')\end{pmatrix}=I$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\omega(t-t') & i\sin\omega(t-t')\\ 0 & i\sin\omega(t-t') & \cos\omega(t-t')\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\omega(t'-t'') & i\sin\omega(t'-t'')\\ 0 & i\sin\omega(t'-t'') & \cos\omega(t'-t'')\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\omega(t-t')\cos\omega(t'-t'')-\sin\omega(t-t')\sin\omega(t'-t'') & i\sin\omega(t-t')\cos\omega(t'-t'')+i\sin\omega(t-t')\cos\omega(t'-t'')\\ 0 & i\sin\omega(t-t')\cos\omega(t'-t'')+i\sin\omega(t-t')\cos\omega(t'-t'') & \cos\omega(t-t')\cos\omega(t'-t'')-\sin\omega(t-t')\sin\omega(t'-t'')\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\omega(t-t'') & i\sin\omega(t-t'')\\ 0 & i\sin\omega(t-t'') & \cos\omega(t-t'')\end{pmatrix}$$

b) Find a Hamiltonian matrix H such that the Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}T(t,t')=HT(t,t')$$

holds as an equality between matrices, and check that for this H it is the case that

$$-i\hbar\frac{\partial}{\partial t'}T(t,t')=HT(t,t')$$

$$i\hbar\begin{pmatrix}0 & 0 & 0\\ 0 & -\omega\sin\omega(t-t') & \omega i\cos\omega(t-t')\\ 0 & \omega i\cos\omega(t-t') & -\omega\sin\omega(t-t')\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\omega(t-t') & -i\sin\omega(t-t')\\ 0 & -i\sin\omega(t-t') & \cos\omega(t-t')\end{pmatrix}$$

$$=i\hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & i\omega\\ 0 & i\omega & 0\end{pmatrix}$$

Similarly,

$$-i\hbar\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\omega(t-t') & -i\sin\omega(t-t')\\ 0 & -i\sin\omega(t-t') & \cos\omega(t-t')\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \omega\sin\omega(t-t') & -\omega i\cos\omega(t-t')\\ 0 & -\omega i\cos\omega(t-t') & \omega\sin\omega(t-t')\end{pmatrix}$$

which on distribution is clearly the same thing.

c) Find the eigenvalues E_n and eigenvectors $|e_n\rangle$ of H, and show that $T(t,t')=\sum_n e^{-i(t-t')\frac{E_n}{\hbar}}|e_n\rangle\langle e_n|$ by calculating the right side as a matrix.

The eigenvalues and eigenvectors of this is

$$0:\begin{bmatrix}1\\0\\0\end{bmatrix}\quad -\hbar w:\begin{bmatrix}0\\-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}\quad \hbar w:\begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

$$T(t,t')=\sum_n e^{-i(t-t')\frac{E_n}{\hbar}}|e_n\rangle\langle e_n|$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}e^{-i(t-t')\frac{1}{\hbar}0} & 0 & 0\\ 0 & \omega e^{-i(t-t')\frac{-i\omega}{\hbar}(-\hbar\omega)} & 0\\ 0 & 0 & \omega e^{-i(t-t')\frac{i\omega}{\hbar}(\hbar\omega)}\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}e^{-i(t-t')\frac{1}{\hbar}0} & 0 & 0\\ 0 & \omega e^{-i(t-t')\frac{-i\omega}{\hbar}(-\hbar\omega)} & 0\\ 0 & 0 & \omega e^{-i(t-t')\frac{i\omega}{\hbar}(\hbar\omega)}\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\omega(t-t') & i\sin\omega(t-t')\\ 0 & i\sin\omega(t-t') & \cos\omega(t-t')\end{bmatrix}$$

4) For two spin-half particles a and b, let

$$|\psi\rangle=\frac{1}{\sqrt{2}}\left(|z_a^+\rangle|z_b^+\rangle+|z_a^-\rangle|z_b^-\rangle\right)$$

a) Show that the properties $[\psi]$ and $[z_a^+]$ are incompatible by evaluation of the commutator $[\psi][z_a^+]-[z_a^+][\psi]$.

$$|\psi\rangle = \frac{1}{2}\left(\left|z_a^+\right\rangle + \left|z_a^-\right\rangle\right)\left(\left\langle z_a^+\right| + \left\langle z_a^-\right|\right)\left|z_a^+\right\rangle\left\langle z_a^+\right| - \frac{1}{2}\left|z_a^+\right\rangle\left\langle z_a^+\right|\left(\left|z_a^+\right\rangle + \left|z_a^-\right\rangle\right)\left(\left\langle z_a^+\right| + \left\langle z_a^-\right|\right)$$

$$= \frac{1}{2}\left(\left|z_a^+\right\rangle\left\langle z_a^+\right| + \left|z_a^-\right\rangle\left\langle z_a^+\right|\right) - \frac{1}{2}\left(\left|z_a^+\right\rangle\left\langle z_a^+\right| + \left|z_a^+\right\rangle\left\langle z_a^-\right|\right) = \frac{1}{2}\left(\left|z_a^-\right\rangle\left\langle z_a^+\right| - \left|z_a^+\right\rangle\left\langle z_a^-\right|\right)$$

I have dropped the b-particle terms entirely since they don't participate in the action with this vector.

b) Show that $[\psi]$ is incompatible with any nontrivial property of spin a, where the trivial properties are I_a and 0.

Note that any operator R on this compound space takes the form
$R = S_{++} \times \left|z_b^+\right\rangle\left\langle z_b^+\right| + S_{+-} \times \left|z_b^+\right\rangle\left\langle z_b^-\right| + S_{-+} \times \left|z_b^+\right\rangle\left\langle z_b^+\right| + S_{--} \times \left|z_b^+\right\rangle\left\langle z_b^+\right|$**.**
Since this property is on a and so is independent of b, the actions of each of these operators S must vanish in order to achieve compatibility.

These operators S then individually have a similar property in that they may also be decomposed into these
$S = A^2\left|z_a^+\right\rangle\left\langle z_a^+\right| + \left(AB^*\right)\left|z_a^+\right\rangle\left\langle z_a^-\right| + \left(BA^*\right)\left|z_a^-\right\rangle\left\langle z_a^+\right| + B^2\left|z_a^-\right\rangle\left\langle z_a^-\right|$, where A, B and are some possibly complex constants. The form is reflective of the fact that this projector must have some consistent form itself.

$$\left[|\psi\rangle\langle\psi|, A\right] = \frac{1}{2}\left(\left|z_a^+\right\rangle + \left|z_a^-\right\rangle\right)\left(\left\langle z_a^+\right| + \left\langle z_a^-\right|\right)\left(A\left|z_a^+\right\rangle\left\langle z_a^+\right| + B\left|z_a^+\right\rangle\left\langle z_a^-\right| + C\left|z_a^-\right\rangle\left\langle z_a^+\right| + D\left|z_a^-\right\rangle\left\langle z_a^-\right|\right)$$

$$-\frac{1}{2}\left(A\left|z_a^+\right\rangle\left\langle z_a^+\right| + B\left|z_a^-\right\rangle\left\langle z_a^+\right| + C\left|z_a^+\right\rangle\left\langle z_a^-\right| + D\left|z_a^-\right\rangle\left\langle z_a^-\right|\right)\left(\left|z_a^+\right\rangle + \left|z_a^-\right\rangle\right)\left(\left\langle z_a^+\right| + \left\langle z_a^-\right|\right)$$

$$= \frac{1}{2}\begin{bmatrix} A\left|z_a^+\right\rangle\left\langle z_a^+\right| + A\left|z_a^-\right\rangle\left\langle z_a^+\right| + B\left|z_a^+\right\rangle\left\langle z_a^-\right| + B\left|z_a^-\right\rangle\left\langle z_a^-\right| \\ + C\left|z_a^-\right\rangle\left\langle z_a^+\right| + C\left|z_a^+\right\rangle\left\langle z_a^+\right| + D\left|z_a^+\right\rangle\left\langle z_a^-\right| + D\left|z_a^-\right\rangle\left\langle z_a^-\right| \end{bmatrix}$$

$$-\frac{1}{2}\begin{bmatrix} A\left|z_a^+\right\rangle\left\langle z_a^+\right| + A\left|z_a^+\right\rangle\left\langle z_a^-\right| + B\left|z_a^+\right\rangle\left\langle z_a^-\right| + B\left|z_a^+\right\rangle\left\langle z_a^+\right| \\ + C\left|z_a^-\right\rangle\left\langle z_a^+\right| + C\left|z_a^-\right\rangle\left\langle z_a^-\right| + D\left|z_a^-\right\rangle\left\langle z_a^-\right| + D\left|z_a^-\right\rangle\left\langle z_a^+\right| \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \left(-B + C\right)\left|z_a^+\right\rangle\left\langle z_a^+\right| + \\ \left(A - D\right)\left|z_a^-\right\rangle\left\langle z_a^+\right| + \\ \left(A - D\right)\left|z_a^+\right\rangle\left\langle z_a^-\right| + \\ \left(B - C\right)\left|z_a^-\right\rangle\left\langle z_a^-\right| \end{bmatrix} = \begin{bmatrix} \left(BA^* - AB^*\right)\left|z_a^+\right\rangle\left\langle z_a^+\right| + \\ \left(A^2 - B^2\right)\left|z_a^-\right\rangle\left\langle z_a^+\right| + \\ \left(A^2 - B^2\right)\left|z_a^+\right\rangle\left\langle z_a^-\right| + \\ \left(AB^* - BA^*\right)\left|z_a^-\right\rangle\left\langle z_a^-\right| \end{bmatrix}$$

So I see by examination the only way that these can all hold is if either both are entirely real and equal (corresponding to I) or all zero, since $BA^* = AB^*$ only for reals.

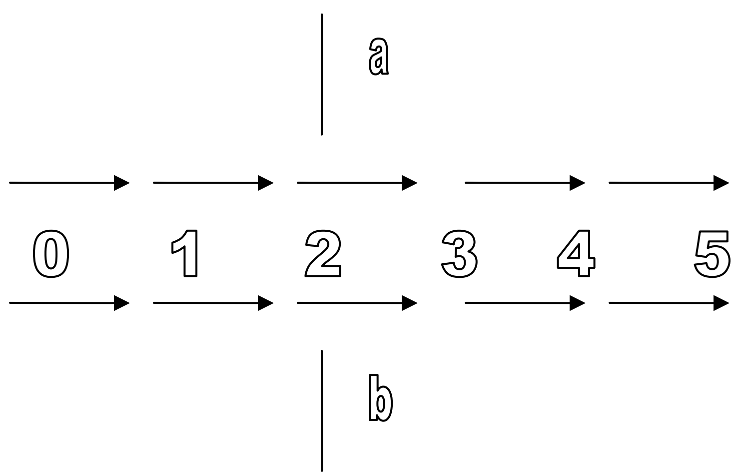
c) **Show that there is no one-dimensional product property $P = A_1 \times B_1$ where A_1 and B_1 are one-dimensional projectors, such that a system in state $|\psi\rangle$ has property P, e.g. $P|\psi\rangle = |\psi\rangle$.**

The proof of this is relatively simple. If such a projector existed, I would be able to decompose $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|z_a^+\rangle|z_b^+\rangle + |z_a^-\rangle|z_b^-\rangle\right)$ into an independent $|z_a^+\rangle$ and $|z_b^+\rangle$ portion. However, no such factorization exists. That is, there is no way to factor a $|z_a^+\rangle$ out of $|\psi\rangle$ and leave behind only $|z_b^+\rangle$ terms.

d) **Find a property $P = A_1 \times B_1 + A_2 \times B_2$ such that $|\psi\rangle$ has property P. What is the physical meaning of P, in words?**

Of course, possibilities exist. For example, take $P = \frac{1}{\sqrt{2}}|z_a^+\rangle\langle z_a^+||z_b^+\rangle\langle z_b^+| + \frac{1}{\sqrt{2}}|z_a^-\rangle\langle z_a^-||z_b^-\rangle\langle z_b^-|$. Here I have simply taken apart the state into the first half and the second half, each of which are perfectly factorizable. Here, $P|\psi\rangle = |\psi\rangle$.

5) **The following toy model is a generalization of the one in Fig 7.1 of Consistent Quantum Theory. A particle (one particle, not two) can be on either the a or be track, so the kets $|mz\rangle$, where z is a or b, and $-M_1 \le m \le M_\mu$, span the particle Hilbert space H_p of dimension 2M, $M = M_l + M_u + 1$. There are two toy detectors with Hilbert spaces H_a, basis $\{|0\hat{a}\rangle, |1\hat{a}\rangle\}$, and H_b, basis $\{|0\hat{b}\rangle, |1\hat{b}\rangle\}$, and the total Hilbert space is $H = H_p \times H_a \times H_b$.**



a) **Write down the time development operator T as a product of the shift operator (with appropriate modifications when m reaches an appropriate**

distance) $S|mz\rangle = |(m+1)z\rangle$**, times other operators which represent the effect of the detectors. Assume the a detector triggers when the particle goes from 2a to 3a, and the b detector triggers when it hops from 2b to 3b; neither detector is affected by the particle moving by on the other track. Be explicit about the form of the other operators multiplying S. How can you be sure that your T is unitary?**

To ensure that T is unitary, the shift operator must eventually "loop around" to the beginning of the particle's tracks.

$$T = S\Big(|1\hat{a}\rangle\big(\langle 1\hat{a}| + \langle 0\hat{a}|\big)|2a\rangle\langle 2a| + |1\hat{b}\rangle\big(\langle 1\hat{b}| + \langle 0\hat{b}|\big)|2b\rangle\langle 2b|\Big)|mz\rangle|a\hat{a}\rangle|b\hat{b}\rangle$$

eventually

$$S|M_\mu z\rangle \to |-1z\rangle$$

b) Work out the unitary time development of the initial state $|\Psi_0\rangle = |\psi_0\rangle \times |0\hat{a}\rangle \times |0\hat{b}\rangle$ **for three different choices of the initial particle state for times up to** $t = M_\mu$**.**

(i) $|\psi_0\rangle = |0a\rangle$

The particle works its way to the right, triggers the a detector on the third motion, and continues on the a track to $t = M_\mu$.

$$|0a\rangle|0\hat{a}\rangle|0\hat{b}\rangle \to |1a\rangle|0\hat{a}\rangle|0\hat{b}\rangle \to |2a\rangle|0\hat{a}\rangle|0\hat{b}\rangle \to |3a\rangle|1\hat{a}\rangle|0\hat{b}\rangle \to |4a\rangle|1\hat{a}\rangle|0\hat{b}\rangle ...$$

(ii) $|\psi_0\rangle = |0b\rangle$

The particle works its way to the right, triggers the b detector on the third motion, and continues on the a track to $t = M_\mu$.

$$|0b\rangle|0\hat{a}\rangle|0\hat{b}\rangle \to |1b\rangle|0\hat{a}\rangle|0\hat{b}\rangle \to |2b\rangle|0\hat{a}\rangle|0\hat{b}\rangle \to |3b\rangle|0\hat{a}\rangle|1\hat{b}\rangle \to |4a\rangle|0\hat{a}\rangle|1\hat{b}\rangle ...$$

(ii) $|\psi_0\rangle = \frac{1}{\sqrt{5}}\big(|0a\rangle + 2|0b\rangle\big)$

The particle works its way to the right, each part "pseudo-triggers" each detector, and it continues.

$$\frac{1}{\sqrt{5}}\big(|0a\rangle + 2|0b\rangle\big)|0\hat{a}\rangle|0\hat{b}\rangle \to \frac{1}{\sqrt{5}}\big(|1a\rangle + 2|1b\rangle\big)|0\hat{a}\rangle|0\hat{b}\rangle$$

$$\to \frac{1}{\sqrt{5}}\big(|2a\rangle + 2|2b\rangle\big)|0\hat{a}\rangle|0\hat{b}\rangle \to \frac{1}{\sqrt{5}}\Big(|3a\rangle|1\hat{a}\rangle|0\hat{b}\rangle + 2|3b\rangle|0\hat{a}\rangle|1\hat{b}\rangle\Big) \to$$

$$\frac{1}{\sqrt{5}}\Big(|4a\rangle|1\hat{a}\rangle|0\hat{b}\rangle + 2|4b\rangle|0\hat{a}\rangle|1\hat{b}\rangle\Big)...$$

c) Use the Born rule in the form $\Pr(P) = \langle\Psi_4|P|\Psi_4\rangle$ **where P is an appropriate projector and** $|\Psi_4\rangle = P^4|\Psi_0\rangle$**, to calculate the probabilities of the following four mutually exclusive probabilities: (i) neither detector has triggered, (ii) detector a but not detector b has triggered, (iii) detector b but not a has**

triggered (iv) both detectors have triggered. Find these probabilities for each of the three cases considered in b.

I simply have taken the $|\Psi_4\rangle$ wave functions from b.

$$P = |0\hat{a}\rangle\langle 0\hat{a}||0\hat{b}\rangle\langle 0\hat{b}| + |1\hat{a}\rangle\langle 1\hat{a}||0\hat{b}\rangle\langle 0\hat{b}| + |0\hat{a}\rangle\langle 0\hat{a}||1\hat{b}\rangle\langle 1\hat{b}| + |1\hat{a}\rangle\langle 1\hat{a}||1\hat{b}\rangle\langle 1\hat{b}|$$

These correspond to the four probabilities desired, and so in the sums below I have included all four terms so as to coincide with these.

(b-i) $\langle\Psi_4|P|\Psi_4\rangle = 0 + 1 + 0 + 0$

(b-ii) $\langle\Psi_4|P|\Psi_4\rangle = 0 + 0 + 1 + 0$

(b-iii) $\langle\Psi_4|P|\Psi_4\rangle = 0 + \frac{1}{5} + \frac{4}{5} + 0$

In other words, based on this the particle either triggered a or b, with corresponding probabilities.

d) Use the Born rule to calculate the conditional probabilities that the particle is at 4a or 4b at t = 4 given that detector a has triggered or detector b has triggered for case b-iii.

Let me decompose the sample space at 3 and 4:

$$P^3 = \left(|0\hat{a}\rangle\langle 0\hat{a}| + |1\hat{a}\rangle\langle 1\hat{a}|\right)\left(|0\hat{b}\rangle\langle 0\hat{b}| + |1\hat{b}\rangle\langle 1\hat{b}|\right)$$

$$P^4 = |4a\rangle\langle 4a| + |4b\rangle\langle 4b|$$

Nothing expressly prohibits both detectors from being triggered, so I leave these spaces open. However, I will select the specific aspects of each that interest me.

$$P(4a\,|\,\hat{a}): |\phi\rangle = [|4a\rangle\langle 4a|]T[|1\hat{a}\rangle\langle 1\hat{a}|]TTT|\psi_0\rangle|0\hat{a}\rangle|0\hat{b}\rangle$$

$$|\phi\rangle = [|4a\rangle\langle 4a|]T[|1\hat{a}\rangle\langle 1\hat{a}|]\frac{1}{\sqrt{5}}\left(|3a\rangle|1\hat{a}\rangle|0\hat{b}\rangle + 2|3b\rangle|0\hat{a}\rangle|1\hat{b}\rangle\right)$$

$$= [|4a\rangle\langle 4a|]T\frac{1}{\sqrt{5}}|3a\rangle|1\hat{a}\rangle|0\hat{b}\rangle = [|4a\rangle\langle 4a|]\frac{1}{\sqrt{5}}|4a\rangle|1\hat{a}\rangle|0\hat{b}\rangle = \frac{1}{\sqrt{5}}|4a\rangle|1\hat{a}\rangle|0\hat{b}\rangle$$

$$P(4a \cap \hat{a}) = \langle\phi||\phi\rangle = \frac{1}{5}$$

$$P(4a\,|\,\hat{b}): |\phi\rangle = [|4a\rangle\langle 4a|]T[|1\hat{b}\rangle\langle 1\hat{b}|]TTT|\psi_0\rangle|0\hat{a}\rangle|0\hat{b}\rangle$$

$$|\phi\rangle = [|4a\rangle\langle 4a|]T[|1\hat{b}\rangle\langle 1\hat{b}|]\frac{1}{\sqrt{5}}\left(|3a\rangle|1\hat{a}\rangle|0\hat{b}\rangle + 2|3b\rangle|0\hat{a}\rangle|1\hat{b}\rangle\right)$$

$$= [|4a\rangle\langle 4a|]T\frac{2}{\sqrt{5}}|3b\rangle|0\hat{a}\rangle|1\hat{b}\rangle = [|4a\rangle\langle 4a|]\frac{1}{\sqrt{5}}|4b\rangle|0\hat{a}\rangle|1\hat{b}\rangle = 0$$

$$P(4a \cap \hat{b}) = \langle\phi||\phi\rangle = 0$$

$$P(4b \mid \hat{a}) : |\phi\rangle = [|4b\rangle\langle 4b|]T[|1\hat{a}\rangle\langle 1\hat{a}|]TTT|\psi_0\rangle|0\hat{a}\rangle|0\hat{b}\rangle$$

$$|\phi\rangle = [|4b\rangle\langle 4b|]T[|1\hat{a}\rangle\langle 1\hat{a}|]\frac{1}{\sqrt{5}}\left(|3a\rangle|1\hat{a}\rangle|0\hat{b}\rangle + 2|3b\rangle|0\hat{a}\rangle|1\hat{b}\rangle\right)$$

$$= [|4b\rangle\langle 4b|]T\frac{1}{\sqrt{5}}|3a\rangle|1\hat{a}\rangle|0\hat{b}\rangle = [|4b\rangle\langle 4b|]\frac{1}{\sqrt{5}}|4a\rangle|1\hat{a}\rangle|0\hat{b}\rangle = 0$$

$$P(4b \cap \hat{a}) = \langle\phi\|\phi\rangle = 0$$

$$P(4b \mid \hat{b}) : |\phi\rangle = [|4b\rangle\langle 4b|]T[|1\hat{b}\rangle\langle 1\hat{b}|]TTT|\psi_0\rangle|0\hat{a}\rangle|0\hat{b}\rangle$$

$$|\phi\rangle = [|4b\rangle\langle 4b|]T[|1\hat{b}\rangle\langle 1\hat{b}|]\frac{1}{\sqrt{5}}\left(|3a\rangle|1\hat{a}\rangle|0\hat{b}\rangle + 2|3b\rangle|0\hat{a}\rangle|1\hat{b}\rangle\right)$$

$$= [|4b\rangle\langle 4b|]T\frac{2}{\sqrt{5}}|3b\rangle|0\hat{a}\rangle|1\hat{b}\rangle = [|4b\rangle\langle 4b|]\frac{1}{\sqrt{5}}|4b\rangle|0\hat{a}\rangle|1\hat{b}\rangle = 0$$

$$P(4\hat{b} \cap \hat{b}) = \langle\phi\|\phi\rangle = \frac{4}{5}$$

By using the usual formula, $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$, it is clear that the conditional probabilities are:

$P(4a \mid \hat{a}) = 1 \quad P(4a \mid \hat{b}) = 0$

$P(4b \mid \hat{b}) = 1 \quad P(4b \mid \hat{a}) = 0$

which yields nothing defying logic.

e) On the basis of your results in c and d, and anything else you consider relevant, address the question of whether a quantum particle can be in two different places at the same time. Are your conclusions limited to the toy model, or do you think they would apply to something more realistic?

Certainly, a particle cannot be in multiple places at the same time. Wave functions that we observe always have norm 1, indicating that they must be at one particular location. The issue at hand is how you ask about behaviors: e.g., a sum of incompatible behaviors may well be compatible, but if you ask about one of the constituents you may not be able to isolate what actually occurred. The toy model, it is important to note, behaves as an automaton, not as the real universe with continuous space and time, and so only replicates the behaviors that we code into it, and here it has replicated what we coded and provided a possible explanation for what we observe, the purpose of any model.