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Quantum Mechanics 2 Homework 1

4 a) For a classical harmonic oscillator with energy:

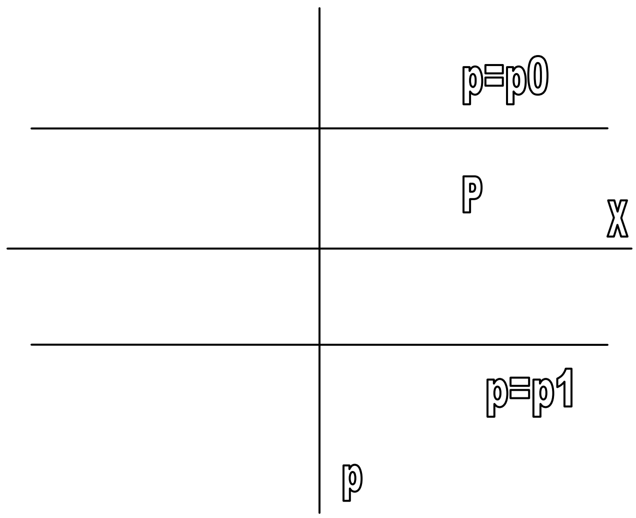
$$E(x,p) = \frac{p^2}{2m} + \frac{1}{2}Kx^2$$

sketch regions in the x, p plane corresponding to the properties

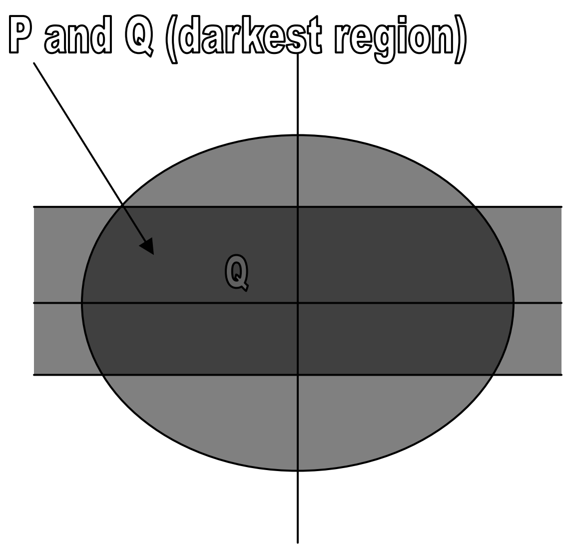
P: $p_0 \le p \le p_1$ **for some** $p_0 < 0 < p_1$

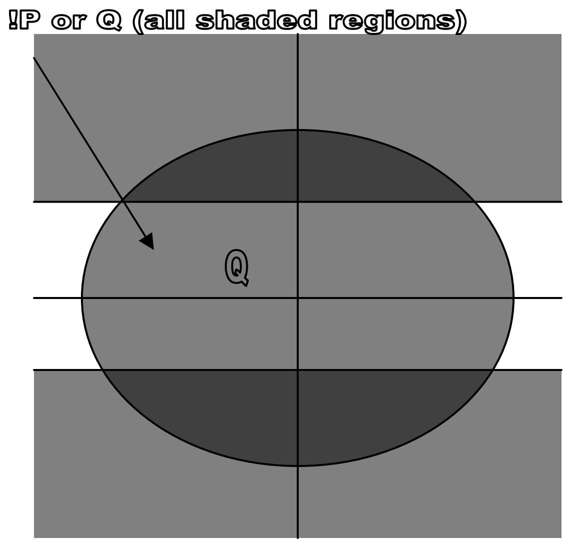
Q: $E \le E_0$ **for some** $E_0 > 0$

The region for condition P is the area between the two horizontal lines:



The region for condition Q is an ellipse, with a peak as high or low as $\sqrt{2mE_0} = \pm p$ and as left or right as $\sqrt{2\frac{E_0}{K}} = \pm x$.





Not P and Q will always be impossible when $p_0 \le \sqrt{2mE_0}$ *and* $p_1 \ge \sqrt{2mE_0}$.

5) **Consider a toy model in one dimension, with $|m\rangle$ the ket for a particle at site m.**

a) **Write down as dyads the of the form $|m\rangle\langle m'|$, or sums of the dyads, the following projectors:**

P: Particle at site m = 1

$|1\rangle\langle 1|$

Q: Particle between 0 and 2. (Q projects onto a three-dimensional subspace).

$|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|$

R: Projector onto the ray (one-dimensional subspace) containing $|\phi\rangle = |1\rangle + 2i|3\rangle$.

$$\frac{1}{5}\left(|1\rangle + 2i|3\rangle\right)\left(\langle 1| - 2i\langle 3|\right)$$

b) **Which of these projectors commute with each other and which do not commute?**

$$PQ - QP = |1\rangle\langle 1|\left(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|\right) - \left(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|\right)|1\rangle\langle 1| = 0$$

$$PR - RP = |1\rangle\langle 1|\left(|1\rangle + 2i|3\rangle\right)\left(\langle 1| - 2i\langle 3|\right) - \left(|1\rangle + 2i|3\rangle\right)\left(\langle 1| - 2i\langle 3|\right)|1\rangle\langle 1|$$

$$= |1\rangle\left(\langle 1| - 2i\langle 3|\right) - \left(|1\rangle + 2i|3\rangle\right)\langle 1| = -2i\left(|1\rangle\langle 3| + |3\rangle\langle 1|\right)$$

Since P and R do not commute and Q has no element in R’s space, Q and R must not commute either.

c) In all cases which two projectors commute, find the projectors corresponding to the conjunction and to the disjunction of the two properties.

$$P \cap Q = |1\rangle\langle 1|$$
$$P \cup Q = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|$$

6) Let $|\phi_n\rangle$ be the state of a harmonic oscillator with energy $E = \left(n + \frac{1}{2}\right)\hbar\omega$, and let $P = [\phi_0] + [\phi_1], Q = [\phi_0] + [\phi_1] + [\phi_2]$ be the projectors for the properties $E < 2\hbar\omega$ and $E < 3\hbar\omega$ respectively.

a) Find the projectors $PQ, \tilde{P}Q, and \quad P\tilde{Q}$ and in each case explain briefly why the property corresponding to the product is what you would expect for the conjunction of the two properties.

$$PQ = [\phi_0] + [\phi_1] + [\phi_2]$$
$$\tilde{P}Q = [\phi_2]$$
$$P\tilde{Q} = 0$$

In the case of PQ, anything less than 3hw must also be less than 2hw necessarily. For $\tilde{P}Q$, I need projectors in Q that are not in P in order to ensure that the energies are less than 3hw and not 2hw. For $P\tilde{Q}$, no projectors can exist such that the energy is less than 2hw and not less than 3hw.

b) Find a nonzero projector R other than $|\phi_0\rangle\langle\phi_0|$, $|\phi_1\rangle\langle\phi_1|$ or P such that PR = R.

R= $Norm(|\phi_0\rangle + |\phi_1\rangle)(\langle\phi_0| + \langle\phi_1|)$

c) Find a projector S such that QS = S but $PS \neq SP$

S= $Norm(|\phi_0\rangle + |\phi_1\rangle + |\phi_2\rangle)(\langle\phi_0| + \langle\phi_1| + \langle\phi_2|)$

$$PS = (|\phi_0\rangle\langle\phi_0| + |\phi_1\rangle\langle\phi_1|)(|\phi_0\rangle + |\phi_1\rangle + |\phi_2\rangle)(\langle\phi_0| + \langle\phi_1| + \langle\phi_2|)$$
$$= (|\phi_0\rangle + |\phi_1\rangle)(\langle\phi_0| + \langle\phi_1| + \langle\phi_2|)$$
$$SP = (|\phi_0\rangle + |\phi_1\rangle + |\phi_2\rangle)(\langle\phi_0| + \langle\phi_1| + \langle\phi_2|)(|\phi_0\rangle\langle\phi_0| + |\phi_1\rangle\langle\phi_1|)$$
$$= (|\phi_0\rangle + |\phi_1\rangle + |\phi_2\rangle)(\langle\phi_0| + \langle\phi_1|) \neq PS$$

7) Show that for a spin-half particle $[z^+]$ and $[x^+]$ do not commute, and then give an argument why the same will be true for the projectors $[v^+]$ and $[w^+]$ for the spin to be along any two directions v and w (unit vectors on the sphere) apart from certain exceptional cases which you should specify.

$$|+_x\rangle = \frac{1}{\sqrt{2}}[|+\rangle + |-\rangle]$$

$$\left[|+\rangle\langle+|,\frac{1}{2}(|+\rangle+|-\rangle)(\langle+|+\langle-|)\right]=\frac{1}{2}|+\rangle\langle+|(|+\rangle+|-\rangle)(\langle+|+\langle-|)-\frac{1}{2}(|+\rangle+|-\rangle)(\langle+|+\langle-|)|+\rangle\langle+|$$

$$=\frac{1}{2}\left[|+\rangle(\langle+|+\langle-|)-(|+\rangle+|-\rangle)\langle+|\right]=\frac{1}{2}\left[|+\rangle\langle-|-|-\rangle\langle+|\right]\neq 0$$

Since the choice of one direction is arbitrary, all that matters is the angle between it and the other vector. The projector must of course commute with itself, and we have seen that it does not commute with the orthogonal projectors. All that's left is to examine the opposite spin, and clearly $\left[|+\rangle\langle+|,|-\rangle\langle-|\right]=0$. Therefore, the spin projectors along the two unit vectors do not commute unless their dot product has magnitude 1.

8) **Find the projectors for the eight events in the Boolean algebra corresponding to the decomposition of the identity** $I = P_1 + P_2 + P_3$**.**

Assuming that these projectors commute, I have

$P_1 \cap P_2 \cap P_3 = P_1 + P_2 + P_3$

$P_1 \cap P_2 \cap \tilde{P}_3 = P_1 + P_2$

$P_1 \cap \tilde{P}_2 \cap P_3 = P_1 + P_3$

$P_1 \cap \tilde{P}_2 \cap \tilde{P}_3 = P_1$

$\tilde{P}_1 \cap P_2 \cap P_3 = P_2 + P_3$

$\tilde{P}_1 \cap P_2 \cap \tilde{P}_3 = P_2$

$\tilde{P}_1 \cap \tilde{P}_2 \cap P_3 = P_3$

$\tilde{P}_1 \cap \tilde{P}_2 \cap \tilde{P}_3 = 0$

9) **The matrices**

$$P=\begin{pmatrix}\frac{1}{6} & \frac{i}{3} & -\frac{i}{6}\\ -\frac{i}{3} & \frac{2}{3} & \frac{-1}{3}\\ \frac{i}{6} & \frac{-1}{3} & \frac{1}{6}\end{pmatrix}\quad Q=\begin{pmatrix}\frac{1}{2} & 0 & -\frac{i}{2}\\ 0 & 1 & 0\\ \frac{i}{2} & 0 & \frac{1}{2}\end{pmatrix}$$

represent two commuting projectors on a three-dimensional Hilbert space H. Find the unique decomposition of the identity I such that P and Q belong to the corresponding Boolean event algebra. Express your answer in terms of 3 by 3 matrices using the same basis as for P and Q above.

Since there are three dimensions in this Hilbert space, there should be only three projectors in the decomposition. Since two are already given, the remaining will be $I-P-Q$, so that $I=P+Q+(I-P-Q)$

$$P=\begin{pmatrix}\frac{1}{6} & \frac{i}{3} & -\frac{i}{6}\\ -\frac{i}{3} & \frac{2}{3} & \frac{-1}{3}\\ \frac{i}{6} & \frac{-1}{3} & \frac{1}{6}\end{pmatrix}\quad Q=\begin{pmatrix}\frac{1}{2} & 0 & -\frac{i}{2}\\ 0 & 1 & 0\\ \frac{i}{2} & 0 & \frac{1}{2}\end{pmatrix}\quad I-P-Q=\begin{pmatrix}\frac{1}{3} & -\frac{i}{3} & \frac{2i}{3}\\ \frac{i}{3} & -\frac{2}{3} & \frac{1}{3}\\ -\frac{2i}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}$$

10) Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two nonzero kets; do not assume they are normalized. Find a simple test for whether or not they represent incompatible properties, using only $|\langle\psi_1|\psi_2\rangle|$, $\langle\psi_1|\psi_1\rangle$, and $\langle\psi_2|\psi_2\rangle$.

Let me construct the test, starting with the definition of compatibility. Note that if $|\langle\psi_1|\psi_2\rangle|$=0, then $\langle\psi_1|\psi_2\rangle=\langle\psi_2|\psi_1\rangle=0$ and the projectors must commute and therefore be compatible by orthogonality. This proof assumes that $|\langle\psi_1|\psi_2\rangle|\neq 0$

$$|\psi_1\rangle\langle\psi_1\|\psi_2\rangle\langle\psi_2|-|\psi_2\rangle\langle\psi_2\|\psi_1\rangle\langle\psi_1|=0$$
$$\langle\psi_1\|\psi_1\rangle\langle\psi_1\|\psi_2\rangle\langle\psi_2|-\langle\psi_1\|\psi_2\rangle\langle\psi_2\|\psi_1\rangle\langle\psi_1|=0$$
$$\langle\psi_1\|\psi_1\rangle\langle\psi_1\|\psi_2\rangle\langle\psi_2\|\psi_2\rangle-\langle\psi_1\|\psi_2\rangle\langle\psi_2\|\psi_1\rangle\langle\psi_1\|\psi_2\rangle=0$$
$$\langle\psi_1\|\psi_1\rangle\langle\psi_2\|\psi_2\rangle-\langle\psi_2\|\psi_1\rangle\langle\psi_1\|\psi_2\rangle=0$$
$$\langle\psi_1\|\psi_1\rangle\langle\psi_2\|\psi_2\rangle-|\langle\psi_2\|\psi_1\rangle|^2=0$$

11) In a particular orthonormal basis the matrix for the observable V is

$$\begin{pmatrix}0 & 1 & i\\ 1 & -\frac{1}{2} & -\frac{i}{2}\\ -i & \frac{i}{2} & -\frac{1}{2}\end{pmatrix}$$

a) Find the corresponding quantum sample space, i.e. decomposition of the identity as a sum of three projectors P_1, P_2, P_3; express each projector as a 3 by 3 matrix in this basis.

Through Jordan decomposition, I find the normalized eigenvectors and get as my projectors:

$$V=\begin{pmatrix}\frac{i}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}}i\\ -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}-2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \sqrt{\frac{2}{3}}i & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{pmatrix}$$

$$I = \frac{1}{3}\begin{pmatrix} 1 & -1 & i \\ -1 & 1 & -i \\ -i & i & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 4 & 2 & -2i \\ 2 & 1 & -i \\ 2i & i & 1 \end{pmatrix}$$

b) Suppose that $\langle V \rangle = 0$, $\langle V^2 \rangle = \frac{6}{5}$. What probabilities should be assigned to each element of the sample space? Be sure and explain what you are doing, then compute $\langle V^3 \rangle$.

In this case,

$$\langle \psi | -2[P_1] + [P_3] \psi \rangle = 0$$

$$-2\langle \psi |[P_1]| \psi \rangle + \langle \psi |[P_3]| \psi \rangle = 0$$

$$\langle \psi |(-2[P_1] + [P_3])(-2[P_1] + [P_3])| \psi \rangle = \frac{6}{5}$$

$$4\langle \psi |[P_1]| \psi \rangle + \langle \psi |[P_3]| \psi \rangle = \frac{6}{5}$$

$$\langle \psi |[P_1]| \psi \rangle = \frac{1}{5} \quad \langle \psi |[P_2]| \psi \rangle = \frac{2}{5} \quad \langle \psi |[P_3]| \psi \rangle = \frac{2}{5}$$

Here I have simply solved the linear system of equations given.

$$\langle V^3 \rangle = -8\langle \psi |[P_1]| \psi \rangle + \langle \psi |[P_3]| \psi \rangle = \frac{-6}{5}$$

c) Here is the matrix of another observable W that commutes with V:

$$\frac{1}{3}\begin{pmatrix} 4 & 5 & 5i \\ 5 & 4 & -5i \\ -5i & 5i & 4 \end{pmatrix}$$

Find the sample space, which in this case consists of two projectors Q_1, Q_2 where Q_1 projects onto a one-dimensional subspace and Q_2 on a two-dimensional space. Assign probabilities of 2/5 and 3/5 to these subspaces respectively, and compute $\langle W \rangle, \langle W^2 \rangle$.

Examining the eigensystem of W, I see that I have the following eigenvalues and normalized eigenvectors:

$$-2 \quad \frac{1}{\sqrt{3}}\begin{bmatrix} -i \\ i \\ 1 \end{bmatrix} \quad 3 \quad \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \quad 3 \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Note that the two corresponding to the duplicated eigenvalue are not orthogonal, and certainly don't need to be. However, in order to produce a useful projector for this space, I'll need to use a Gram-Schmidt style method in the second projector. Hypothetically, I could also just take the cross product of the other two vectors normalized.

$$Q_1 = |e_1\rangle\langle e_1| = \frac{1}{3}\begin{bmatrix} 1 & -1 & -i \\ -1 & 1 & i \\ i & -i & 1 \end{bmatrix}$$

$$Q_2 = |e_1\rangle\langle e_1| + \frac{1}{\left\| \langle e_2|e_2\rangle - \left\| \langle e_1|e_2\rangle \right\|^2 \langle e_1|e_1\rangle \right\|} \left(|e_2\rangle - \langle e_1|e_2\rangle |e_1\rangle\right)\left(\langle e_2| - \langle e_1|\langle e_2|e_1\rangle\right)$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 & -i \\ 1 & 2 & i \\ i & -i & 2 \end{bmatrix}$$

$$I = Q_1 + Q_2$$

$$Q = -2Q_1 + 3Q_2$$

$$\langle Q \rangle = -2\left(\frac{2}{5}\right) + 3\left(\frac{3}{5}\right) = 1$$

$$\langle Q^2 \rangle = 4\left(\frac{2}{5}\right) + 9\left(\frac{3}{5}\right) = 7$$

d) Even if the probability distribution is not given in advance, it is possible to sensibly discuss certain assertions of the form, "If W = b, then the probability that V = a is…" where "…" might be "one", "zero", or "unknown". Which framework (sample space) should you use in order to discuss assertions of this form in a meaningful way? Work out examples of sensible statements of this type for various choices of b and a.

Since V and W commute, VW = WV and it makes sense to consider questions of this variety. By using the sample space $W \times V$, I can examine assertions of this type. Then, using the identity for the decomposition I have

$$\langle \psi | ([V_1] + [V_2] + [V_3])([Q_1] + [Q_2]) | \psi \rangle$$

now by choosing a value and therefore a projector of W, for example 3, I get

$$\langle \psi | ([V_1] + [V_2] + [V_3])([Q_2]) | \psi \rangle$$

and collapsing the kets in each case, I have

$\langle \psi | [V_1][Q_2] | \psi \rangle$, which would give the relative probabilities of each V result which may well be one if $[V_1][Q_2] = 1$, zero if $[V_1][Q_2] = 0$ or unknown otherwise.