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Practice for Qualifying Exams

Problem Source: **CMU February 2006 Qualifying Exam**

A particle of mass m is in a 1-d harmonic oscillator potential

$$V = \frac{1}{2} m\omega^2 x^2$$

(a) Show that the Hamiltonian can be written as

$$H = \hbar\omega\left(a^+ a + \frac{1}{2}\right)$$

where a, a^+ **are the lowering and raising operators, respectively. Write these operators in terms of x and p.**

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 + (m\omega x)^2\right]$$

write

$$\frac{1}{2m}\left[\frac{\hbar}{i}\frac{\partial}{\partial x} + im\omega x\right]\left[\frac{\hbar}{i}\frac{\partial}{\partial x} - im\omega x\right] = \frac{1}{2m}\left[-\hbar^2\frac{\partial^2}{\partial x^2} - \hbar m\omega\left[1 + x\frac{\partial}{\partial x}\right] + \hbar m\omega x\frac{\partial}{\partial x} + (m\omega x)^2\right]$$

$$= \frac{1}{2m}\left[-\hbar^2\frac{\partial^2}{\partial x^2} - \hbar m\omega + (m\omega x)^2\right] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 - \frac{\hbar\omega}{2} = H - \frac{\hbar\omega}{2}$$

then

$$\frac{1}{2m}\left[\frac{\hbar}{i}\frac{\partial}{\partial x} + im\omega x\right]\left[\frac{\hbar}{i}\frac{\partial}{\partial x} - im\omega x\right] = a^+ a$$

$$= \hbar\omega\left[\frac{1}{\sqrt{2m\hbar\omega}}\left(\frac{\hbar}{i}\frac{\partial}{\partial x} + im\omega x\right)\right]\left[\frac{1}{\sqrt{2m\hbar\omega}}\left(\frac{\hbar}{i}\frac{\partial}{\partial x} - im\omega x\right)\right]$$

$$= \hbar\omega\left[\frac{1}{\sqrt{2m\hbar\omega}}(p + im\omega x)\right]\left[\frac{1}{\sqrt{2m\hbar\omega}}(p - im\omega x)\right]$$

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(p - im\omega x) \quad a^+ = \frac{1}{\sqrt{2m\hbar\omega}}(p + im\omega x)$$

Note that I have chosen an unusual convention here, where p is real and x is imaginary. I could just as well multiply through by 1 in the form of i and –i.

(b) Show that:

$$a^+|n\rangle = \alpha_+(n)|n+1\rangle \quad a|n\rangle = \alpha_-(n)|n-1\rangle$$

and evaluate $\alpha_+(n)$ and $\alpha_-(n)$. You can assume that the eigenvalues of the Hamiltonian are $\hbar\omega\left(n+\frac{1}{2}\right)$ with eigenvectors $|n\rangle$, with n an integer greater than zero.

First, note that: $H = \hbar\omega\left(a^+a+\frac{1}{2}\right)$, but suppose I had chosen aa^+ instead:

$$aa^+ = \frac{1}{2m\hbar\omega}[p-im\omega x][p+im\omega x] = \frac{1}{2m\hbar\omega}\left[\frac{\hbar}{i}\frac{\partial}{\partial x}-im\omega x\right]\left[\frac{\hbar}{i}\frac{\partial}{\partial x}+im\omega x\right]$$
$$= \frac{1}{2m\hbar\omega}\left[-\hbar^2\frac{\partial^2}{\partial x^2}+\omega^2 m^2 x^2+\hbar m\omega\left[1+x\frac{\partial}{\partial x}\right]-\hbar m\omega x\frac{\partial}{\partial x}\right]$$
$$= \frac{1}{2m\hbar\omega}\left[-\hbar^2\frac{\partial^2}{\partial x^2}+\omega^2 m^2 x^2+\hbar m\omega\right]$$
$$H = \hbar\omega\left(aa^+-\frac{1}{2}\right)$$

Now I see that:

$$H|n\rangle = \hbar\omega\left(a^+a+\frac{1}{2}\right)|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle$$
$$a\hbar\omega\left(a^+a+\frac{1}{2}\right)|n\rangle = a\hbar\omega\left(n+\frac{1}{2}\right)|n\rangle$$
$$\hbar\omega\left(aa^++\frac{1}{2}\right)a|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)a|n\rangle$$
$$\hbar\omega\left(aa^+-\frac{1}{2}\right)a|n\rangle + \hbar\omega a|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)a|n\rangle$$
$$\hbar\omega\left(aa^+-\frac{1}{2}\right)a|n\rangle = \hbar\omega\left(n-\frac{1}{2}\right)a|n\rangle$$
$$H\big[a|n\rangle\big] = \hbar\omega\left(n-\frac{1}{2}\right)\big[a|n\rangle\big] = H|n-1\rangle$$

clearly, then:

$$a|n\rangle = \sqrt{n}|n-1\rangle \rightarrow \alpha_-(n) = \sqrt{n}$$

and this proof works similarly for the other form of the Hamiltonian, giving

$$H|n\rangle = \hbar\omega\left(a^+ a + \frac{1}{2}\right)|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$$

$$a\hbar\omega\left(a^+ a + \frac{1}{2}\right)|n\rangle = a\hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$$

$$\hbar\omega\left(aa^+ + \frac{1}{2}\right)a|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)a|n\rangle$$

$$\hbar\omega\left(aa^+ - \frac{1}{2}\right)a|n\rangle + \hbar\omega a|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)a|n\rangle$$

$$\hbar\omega\left(aa^+ - \frac{1}{2}\right)a|n\rangle = \hbar\omega\left(n - \frac{1}{2}\right)a|n\rangle$$

$$H\left[a^+|n\rangle\right] = \hbar\omega\left(n + \frac{3}{2}\right)\left[a^+|n\rangle\right] = H|n+1\rangle$$

clearly, then:

$$a^+|n\rangle = \sqrt{n+1}|n+1\rangle \to \alpha_+(n) = \sqrt{n+1}$$

(c) Now consider adding to the system another distinguishable particle of equal mass whose potential is given by

$$V(x_2) = \frac{1}{2}m\omega^2 x_2^2$$

The particles are coupled via a perturbing force

$$V_{12} = \frac{1}{4}m\Omega^2(x_1 - x_2)^2$$

Use perturbation theory to calculate the first order correction to the ground state energy of the system.

First note that:

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(p - im\omega x) \quad a^+ = \frac{1}{\sqrt{2m\hbar\omega}}(p + im\omega x)$$

$$x = -i\frac{\sqrt{2m\hbar\omega}}{2m\omega}(a^+ - a) = i\sqrt{\frac{\hbar}{2m\omega}}(a - a^+)$$

$$\langle 0_1|\langle 0_2|V_{12}|0_1\rangle|0_2\rangle = \frac{1}{4}m\Omega^2\langle 0_1|\langle 0_2|x_1^2 + x_2^2 - x_1x_2 - x_2x_1|0_1\rangle|0_2\rangle$$

$$[x_1, x_2] = 0$$

$$= -\frac{1}{4}\frac{\hbar}{2\omega}\Omega^2\langle 0_1|\langle 0_2|(a_1 - a_1^+)^2 + (a_2 - a_2^+)^2 - 2(a_1 - a_1^+)(a_2 - a_2^+)|0_1\rangle|0_2\rangle$$

Now note that:

$$\left(a-a^{+}\right)|n\rangle=\sqrt{n}|n-1\rangle-\sqrt{n+1}|n+1\rangle$$

$$\left(a-a^{+}\right)^{2}|n\rangle=\sqrt{n}\sqrt{n-1}|n-2\rangle-(n+1)|n\rangle-n|n\rangle+\sqrt{n+1}\sqrt{n+2}|n+2\rangle$$

So that applying these formulae and eliminating orthogonal kets, I have

$$=-\frac{1}{4}\frac{\hbar}{2\omega}\Omega^{2}\langle 0_{1}|\langle 0_{2}|(-2)|0_{1}\rangle|0_{2}\rangle=\frac{1}{4}\frac{\hbar}{\omega}\Omega^{2}$$

as the first order correction.

(d) Use first-order perturbation theory to calculate the energies of the first two excited states. What are the corresponding eigenvectors?

I will need the matrix elements

$$\begin{bmatrix}\langle 0_{1}|\langle 1_{2}|V_{12}|0_{1}\rangle|1_{2}\rangle & \langle 0_{1}|\langle 1_{2}|V_{12}|1_{1}\rangle|0_{2}\rangle\\ \langle 1_{1}|\langle 0_{2}|V_{12}|0_{1}\rangle|1_{2}\rangle & \langle 1_{1}|\langle 0_{2}|V_{12}|1_{1}\rangle|0_{2}\rangle\end{bmatrix}$$

Certainly,

$$\langle 0_{1}|\langle 1_{2}|V_{12}|0_{1}\rangle|1_{2}\rangle=\langle 1_{1}|\langle 0_{2}|V_{12}|1_{1}\rangle|0_{2}\rangle$$

$$\langle 0_{1}|\langle 1_{2}|V_{12}|1_{1}\rangle|0_{2}\rangle=\left[\langle 1_{1}|\langle 0_{2}|V_{12}|0_{1}\rangle|1_{2}\rangle\right]^{+}$$

Using the identities above,

$$\langle 0_{1}|\langle 1_{2}|V_{12}|0_{1}\rangle|1_{2}\rangle=\frac{1}{4}m\Omega^{2}\langle 0_{1}|\langle 1_{2}|x_{1}^{2}+x_{2}^{2}-x_{1}x_{2}-x_{2}x_{1}|0_{1}\rangle|1_{2}\rangle$$

$$=-\frac{1}{4}\frac{\hbar}{2\omega}\Omega^{2}\langle 0_{1}|\langle 1_{2}|\left(a_{1}-a_{1}^{+}\right)^{2}+\left(a_{2}-a_{2}^{+}\right)^{2}-2\left(a_{1}-a_{1}^{+}\right)\left(a_{2}-a_{2}^{+}\right)|0_{1}\rangle|1_{2}\rangle$$

$$=-\frac{1}{4}\frac{\hbar}{2\omega}\Omega^{2}\langle 0_{1}|\langle 1_{2}|[-1-3]|0_{1}\rangle|1_{2}\rangle=\frac{1}{2}\frac{\hbar}{\omega}\Omega^{2}$$

$$\langle 0_{1}|\langle 1_{2}|V_{12}|1_{1}\rangle|0_{2}\rangle=\frac{1}{4}m\Omega^{2}\langle 0_{1}|\langle 1_{2}|x_{1}^{2}+x_{2}^{2}-x_{1}x_{2}-x_{2}x_{1}|1_{1}\rangle|0_{2}\rangle$$

$$=-\frac{1}{4}\frac{\hbar}{2\omega}\Omega^{2}\langle 0_{1}|\langle 1_{2}|\left(a_{1}-a_{1}^{+}\right)^{2}+\left(a_{2}-a_{2}^{+}\right)^{2}-2\left(a_{1}-a_{1}^{+}\right)\left(a_{2}-a_{2}^{+}\right)|1_{1}\rangle|0_{2}\rangle$$

$$=-\frac{1}{4}\frac{\hbar}{2\omega}\Omega^{2}\langle 0_{1}|\langle 1_{2}|2|1_{1}\rangle|0_{2}\rangle=-\frac{1}{4}\frac{\hbar}{\omega}\Omega^{2}$$

So my matrix is then

$$\frac{1}{4}\frac{\hbar}{\omega}\Omega^{2}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$$

The eigenvalues are then given by

$$(2-\lambda)^2 - 1 = 0$$
$$\lambda^2 - 4\lambda + 3 = 0$$
$$(\lambda - 1)(\lambda - 3) = 0$$
$$\lambda \in \{1,3\}$$

So the energies are then

$$\frac{1}{4}\frac{\hbar}{\omega}\Omega^2, \frac{3}{4}\frac{\hbar}{\omega}\Omega^2$$

The eigenvectors of these, normalized, are then

$$1: Null\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = |-\rangle = \frac{1}{\sqrt{2}}|0_1\rangle|1_2\rangle - \frac{1}{\sqrt{2}}|1_1\rangle|0_2\rangle$$
$$3: Null\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle = \frac{1}{\sqrt{2}}|0_1\rangle|1_2\rangle + \frac{1}{\sqrt{2}}|1_1\rangle|0_2\rangle$$

Which correspond to the “plus” and “minus” combinations of the degenerate first excited states.

(e) At t = 0 the system is prepared such that particle one is in the state $|1\rangle$ and particle two is in the state $|0\rangle$. What is the probability of finding particle one in the unperturbed state $|1\rangle$ as a function of time?

Note: $|1_1\rangle|0_2\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$ (using the notation from part d).

$$\rightarrow \frac{1}{\sqrt{2}}\left(e^{-i\left(\frac{\omega}{2}+\frac{3\Omega^2}{4\omega}\right)t}\left|+\right\rangle + e^{-i\left(\frac{\omega}{2}+\frac{\Omega^2}{4\omega}\right)t}\left|-\right\rangle\right)$$

$$= \left|\left\langle 1_1\right|\left\langle 0_2\right|\frac{1}{\sqrt{2}}\left[e^{-i\left(\frac{\omega}{2}+\frac{3\Omega^2}{4\omega}\right)t}\left|+\right\rangle + e^{-i\left(\frac{\omega}{2}+\frac{\Omega^2}{4\omega}\right)t}\left|-\right\rangle\right]\right|^2 + \left|\left\langle 1_1\right|\left\langle 1_2\right|\frac{1}{\sqrt{2}}\left[e^{-i\left(\frac{\omega}{2}+\frac{3\Omega^2}{4\omega}\right)t}\left|+\right\rangle + e^{-i\left(\frac{\omega}{2}+\frac{\Omega^2}{4\omega}\right)t}\left|-\right\rangle\right]\right|^2$$

$$= \left|\left\langle 1_1\right|\left\langle 0_2\right|\frac{1}{\sqrt{2}}\left[e^{-i\left(\frac{\omega}{2}+\frac{3\Omega^2}{4\omega}\right)t}\left|+\right\rangle + e^{-i\left(\frac{\omega}{2}+\frac{\Omega^2}{4\omega}\right)t}\left|-\right\rangle\right]\right|^2 + 0$$

$$= \frac{1}{4}\left|\left[e^{-i\left(\frac{\omega}{2}+\frac{3\Omega^2}{4\omega}\right)t} - e^{-i\left(\frac{\omega}{2}+\frac{\Omega^2}{4\omega}\right)t}\right]\right|^2$$

$$= \frac{1}{4}\left[e^{-i\left(\frac{\omega}{2}+\frac{3\Omega^2}{4\omega}\right)t} - e^{-i\left(\frac{\omega}{2}+\frac{\Omega^2}{4\omega}\right)t}\right]\left[e^{i\left(\frac{\omega}{2}+\frac{3\Omega^2}{4\omega}\right)t} - e^{i\left(\frac{\omega}{2}+\frac{\Omega^2}{4\omega}\right)t}\right] = \frac{1}{4}\left(2 - e^{-i\frac{\Omega^2}{2\omega}t} - e^{i\frac{\Omega^2}{2\omega}t}\right)$$

So I see that the result fluctuates between zero and one, just as one might expect.

Thanks to LC for correcting an error in this problem.