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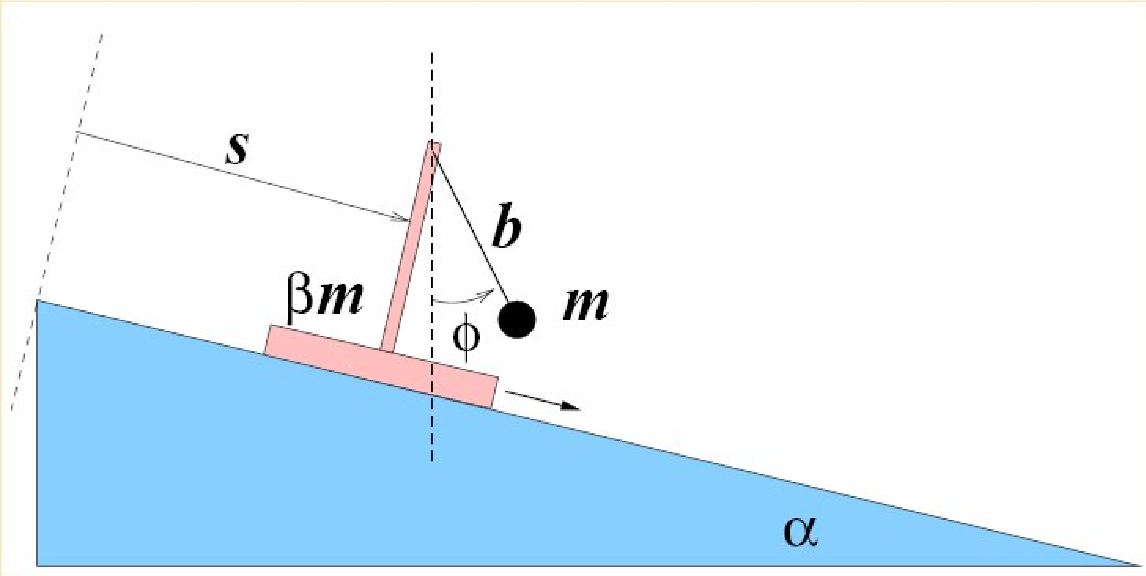
Practice for Qualifying Exams

Problem Source: **CMU Qualification Exam Day 1 (February 2006)**



(1) A pendulum of mass m and length b is mounted to a heavy sled of mass βm which is free to slide with negligible friction down an inclined plane of slope α as shown. The entire system acts under the influence of gravity, where g is the downward acceleration due to gravity.

(a) Let ϕ denote the angle that the pendulum bob makes with the vertical and s denote the distance that the sled has slid down the slope (see diagram). Determine the Lagrangian $L(s,\phi,\dot{s},\dot{\phi})$ describing the system in terms of this angle and distance and their time derivatives.

The terms to consider are:

-The kinetic energy of the main body of the sled:

$$K_{body} = \frac{1}{2}\beta m\dot{s}^2$$

-The kinetic energy of the pendulum bob:

$$K_{bob} = \frac{1}{2}m\left[\left(b\dot{\phi}\cos(\phi+\alpha)+\dot{s}\right)^2 + \left(b\dot{\phi}\sin(\phi+\alpha)\right)^2\right] = \frac{1}{2}m\left[2b\dot{\phi}\dot{s}\cos(\phi+\alpha)+\dot{s}^2+\left(b\dot{\phi}\right)^2\right]$$

-The potential energy of the main body of the sled:

$$V_{body} = \beta mg(-s\sin\alpha) + C_{1,geometry}$$

The constant does not factor into any equation of motion: it simply represents that this "center of mass" potential does not take into account any rearrangement of the orientation of the sled.

-The potential energy of the pendulum bob

$$V_{bob} = mg(-s\sin\alpha) + mgb(1-\cos\phi) + C_{2,\,geometry}$$

Again, the constant represents some constant of geometry like the height of the mast from which the bob hangs—this does not factor into any equation of motion.

The relevant part of the Lagrangian, then, is:

$$L = K - V = \frac{1}{2}\beta m\dot{s}^2 + \frac{1}{2}m\left(2b\dot{\phi}\dot{s}\cos(\phi+\alpha) + \dot{s}^2 + \left(b\dot{\phi}\right)^2\right) - \beta mg(-s\sin\alpha) - mg(-s\sin\alpha) - mgb(1-\cos\phi)$$

(b) Find expressions for the conjugate momenta p_s, p_ϕ of s, ϕ, respectively.

$$p_s = \frac{\partial L}{\partial \dot{s}}$$

$$p_s = \beta m\dot{s} + \frac{1}{2}m\left(2b\dot{\phi}\cos(\phi+\alpha) + 2\dot{s}\right)$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}}$$

$$p_\phi = \frac{1}{2}m\left(2b\dot{s}\cos(\phi+\alpha) + 2b^2\dot{\phi}\right)$$

(c) Describe the steps you would use to find the Hamiltonian $H(\phi, s, p_\phi, p_s)$ of the system. Do not actually compute H.

The Hamiltonian is given by $H = \sum_i q_i p_i - L = sp_s + \phi p_\phi - L$. The only step necessary to obtain the form $H(\phi, s, p_\phi, p_s)$, then, is to solve the expressions for conjugate momenta in terms of the time-derivatives of the coordinates and substitute these into the Lagrangian above.

(d) From the equations of motion for the system, find expressions for the generalized accelerations $\ddot{\phi}$ and $\ddot{s}$ in terms of $s, \phi, \dot{s}, \dot{\phi}$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right)=\frac{\partial L}{\partial s}$$

$$\frac{d}{dt}\left[\beta m\dot{s}+\frac{1}{2}m\left(2b\dot{\phi}\cos(\phi+\alpha)+2\dot{s}\right)\right]=-\beta mg(-\sin\alpha)-mg(-\sin\alpha)$$

$$\left[\beta m\ddot{s}+\frac{1}{2}m\left(2b\ddot{\phi}\cos(\phi+\alpha)-2b\dot{\phi}^2\sin(\phi+\alpha)+2\ddot{s}\right)\right]=\beta mg(\sin\alpha)+mg(\sin\alpha)$$

$$\underline{\underline{\left[(\beta+1)m\ddot{s}+mb\left(\ddot{\phi}\cos(\phi+\alpha)-\dot{\phi}^2\sin(\phi+\alpha)\right)\right]=(\beta+1)mg(\sin\alpha)}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)=\frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt}\left(\frac{1}{2}m\left(2b\dot{s}\cos(\phi+\alpha)+2b^2\dot{\phi}\right)\right)=-\frac{1}{2}m\left(2b\dot{\phi}\dot{s}\sin(\phi+\alpha)\right)-mgb\sin\phi$$

$$\frac{1}{2}m\left(2b\ddot{s}\cos(\phi+\alpha)-2b\dot{s}\dot{\phi}\sin(\phi+\alpha)+2b^2\ddot{\phi}\right)=-\frac{1}{2}m\left(2b\dot{\phi}\dot{s}\sin(\phi+\alpha)\right)-mgb\sin\phi$$

$$\underline{\underline{mb\left(\ddot{s}\cos(\phi+\alpha)+b\ddot{\phi}\right)=-mgb\sin\phi}}$$

(e) The sled can slide down the slope with the pendulum at an unchanging angle ϕ_0 if ϕ_0 satisfies a certain condition. What is this condition?

Take the above equations of motion with $\phi\to\phi_0,\dot{\phi}\to 0,\ddot{\phi}\to 0$

$$(\beta+1)m\ddot{s}=(\beta+1)mg(\sin\alpha)$$

$$mb\ddot{s}\cos(\phi+\alpha)=-mgb\sin\phi_0$$

and

$$\ddot{s}=g(\sin\alpha)$$

$$\ddot{s}\cos(\phi+\alpha)=-g\sin\phi_0$$

$$\sin\alpha\cos(\phi+\alpha)+\sin\phi_0=0$$