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Practice for Qualifying Exams

Problem Source: **CMU February 2006 Qualifying Exam**

This problem has to do with diffusion of heat in a slab of material. For the purposes of this problem we will ignore thermal expansion. As such, we may define the internal energy density in terms of the heat capacity per unit mass c_p such that the total energy of the sample is given by:

$$E = \int d^3 r c_p \rho(\vec{r},t) T(\vec{r},t)$$

where ρ is the mass density and T is the temperature. The "heat current density" is defined by

$$\vec{j} = -k_{th} \vec{\nabla} T(\vec{r},t)$$

where k_{th} is the "thermal conductivity".

(a) Assuming that there are no sources or sinks and that the density is constant derive the heat diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

expressing the diffusivity constant κ in terms of c_p, k_{th}, ρ. What are the units of κ?

The heart of each diffusion equation comes from the assumption that there are no sources or sinks of energy: namely, that

$$\frac{\partial E}{\partial t} = -\vec{\nabla} \cdot \vec{q}$$

Note that in this material, $\vec{q} = c_p \rho \vec{j} = -c_p \rho k_{th} \vec{\nabla} T(\vec{r},t)$

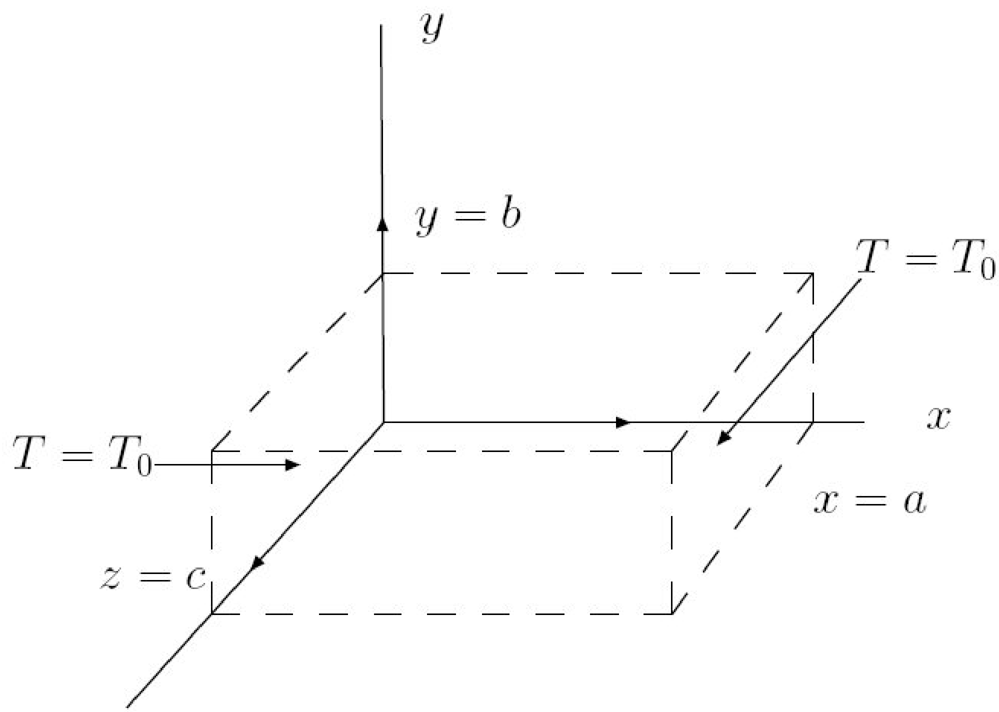
Treating these as local relations, then, I have

$$c_p \rho \frac{\partial T}{\partial t} = c_p \rho k_{th} \nabla^2 T$$

$$\frac{\partial T}{\partial t} = k_{th} \nabla^2 T$$

so that κ has units $\frac{m^2}{s}$.

(b) Consider a slab of material with diffusivity κ that has a uniform mass distribution $\rho = \rho_0$. The geometry is such that at x = 0 and x = a the slab is held at fixed temperature T_0. The remaining sides are insulated (no heat flow in or out). The situation is depicted in the picture shown below:



At t = 0, the temperature is given by $T(\vec{r},0)$. What are the appropriate spatial boundary conditions?

$T(y\hat{y} + z\hat{z},0) = T_0 \quad T(a\hat{x} + y\hat{y} + z\hat{z},0) = T_0$

$\vec{\nabla}T(x\hat{x} + y\hat{y},0)\cdot\hat{z} = 0 \quad \vec{\nabla}T(x\hat{x} + y\hat{y} + c\hat{z},0)\cdot\hat{z} = 0 \quad \vec{\nabla}T(x\hat{x} + z\hat{z},0)\cdot\hat{y} = 0 \quad \vec{\nabla}T(x\hat{x} + z\hat{z} + b\hat{y},0)\cdot\hat{y} = 0$

(c) To solve for the temperature as a function of time it is best to write the diffusion equation in terms of the variable

$$\delta T = T - T_0$$

such that when $t \to \infty$ we have $\delta T \to 0$. Using the ansatz

$$\delta T = \delta T(\vec{r})e^{-\lambda t}$$

derive an equation of the form:

$$\left(\nabla^2 + \frac{\lambda}{\kappa}\right)\delta T(\vec{r}) = 0$$

and write down an expression (in terms of an infinite sum) for all possible values of λ.

Using

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

I have

$$\frac{\partial}{\partial t}(T_0 + \delta T) = \kappa \nabla^2 (T_0 + \delta T)$$
$$\lambda \delta T(\vec{r}) e^{-\lambda t} = \kappa \nabla^2 \delta T(\vec{r}) e^{-\lambda t}$$
$$-\frac{\lambda}{\kappa}\delta T(\vec{r}) = \nabla^2 \delta T(\vec{r})$$
$$\left(\nabla^2 + \frac{\lambda}{\kappa}\right)\delta T(\vec{r}) = 0$$

I see then that $-\frac{\lambda}{\kappa}$ must be a sum of eigenvalues of ∇^2 on $\delta T(\vec{r})$.

In this case, the eigenfunctions will be $\delta T(\vec{r}) = \sin\left(\frac{j\pi}{a}x\right)\cos\left(\frac{k\pi}{b}y\right)\cos\left(\frac{l\pi}{c}z\right)$ for integers j, k and l. In this case, then, the eigenvalues are $-\left(\frac{j\pi}{a}\right)^2 - \left(\frac{k\pi}{b}\right)^2 - \left(\frac{l\pi}{c}\right)^2$, and so $\lambda \in \bigcup_{j,k,l=0}^{\infty}\left[\left(\frac{j\pi}{a}\right)^2 + \left(\frac{k\pi}{a}\right)^2 + \left(\frac{l\pi}{a}\right)^2\right]$, and the function itself may be a sum of these eigenfunctions.

(d) Show that in the long-time limit only one term of all the possible values in the above sum is relevant and determine that value.

Given the decay exponential, I see that only the smallest non-zero coefficient will matter in the long-time limit: namely, in this case, $\lambda = \kappa\left(\frac{\pi}{a}\right)^2$, since the cosine values may be zero and still allow δT to be nonzero.

(e) Write down the time dependent expression for the temperature in terms of the initial distribution $T(\vec{r},0)$ in the long time limit (keeping the first derivation from the infinite time limit $T = T_0$). The final expression should have an integral remaining to be done over $T(\vec{r},0)$.

Projecting the initial distribution into the eigenspace (normalization will drop out in the end and so is not considered), I have:

$$C = \int d^3r\left[T(\vec{r},0) - T_0\right]\sin\left(\frac{\pi}{a}x\right)$$

$$T(\vec{r},t) = T_0 + C\sin\left(\frac{\pi}{a}x\right)e^{-\kappa\left(\frac{\pi}{a}\right)^2 t}$$