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Practice for Qualifying Exams

Thanks to Ryan for his input in this problem.

<u>Problem Source: **CMU February 2006 Qualifying Exam**</u>

Consider an azimuthally symmetric magnetic field, such as might be produced by a dipole magnet with circular pole faces, of the form $\vec{B} = B(r)\hat{z}$. An electron of charge e orbits in this field at a fixed radius R with momentum $\vec{p}$ in the (r, θ) plane.

(a) What is the frequency of the electron, ω_0, assuming the magnetic field is time independent?

$$\vec{F}_B = q\vec{v} \times \vec{B} = -e\frac{p}{m}B(R)\hat{r}$$

$$\vec{F}_c = -m\frac{v^2}{R}\hat{r} = -\frac{p^2}{mR}\hat{r}$$

$$\frac{p^2}{mR} = e\frac{p}{m}B(R)$$

$$p = eRB(R)$$

$$v = \frac{eR}{m}B(R)$$

$$\omega_0 = \frac{2\pi R}{v} = \frac{2\pi m}{e}B(R)$$

(b) Suppose we inductively accelerate this electron by varying the magnetic field slowly in time in some way. We want the electron's orbit to remain fixed at R. Derive a condition that relates $B(r)$ and the average value of the field within the orbit, $\langle B \rangle$, that allows this to happen. This is the so-called Betatron condition. Make a rough sketch of $B(r)$ vs. r that allows this acceleration technique to work.

The fact that we are working with the average value of the field inside the orbit seems to imply that Faraday's Law would be a good place to start:

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}(r) \cdot \hat{n}dS$$

$$\langle B \rangle = \frac{1}{\pi R^2}\int_S \vec{B}(r) \cdot \hat{n}dS$$

$$\oint_C \vec{E} \cdot d\vec{l} = -\pi R^2 \frac{d}{dt}\langle B \rangle$$

By symmetry, the electric field must be the same all the way around the loop. Then, I have:

$$-\pi R^2 \frac{d}{dt}\langle B\rangle = \oint_C \vec{E}\cdot d\vec{l} = 2\pi R E(R)$$

However, I certainly know from Gauss's law and charge-free space, that $\vec{E}(r) = E(r)\hat{\phi}$. Now I have that:

$$-eE(R) = m\frac{dv}{dt}$$

and from above,

$$v = \frac{eR}{m}B(R)$$

so that for circular orbits,

$$\frac{dv}{dt} = \frac{eR}{m}\frac{dB(R)}{dt}$$

combining, then, I have that

$$-\pi R^2 \frac{d\langle B\rangle}{dt} = 2\pi R E(R) = -2\pi R\left[R\frac{dB(r)}{dt}\right]$$

$$\frac{d\langle B\rangle}{dt} = 2\frac{dB(R)}{dt}$$

So with the average value of the magnetic field inside of a circle being double what it is in the border, I expect to see roughly a mound-shaped profile.

(c) What is the change in momentum if the magnetic field scale starts at B_0 and ends at B_1?

In this case, then,

$$\Delta\langle B\rangle = B_1 - B_0 = 2\Delta B(R)$$

$$\Delta B(R) = \frac{B_1 - B_0}{2}$$

Now assuming that the radius of orbit stays constant, I may evaluate:

$$p_0 = eRB(R)$$
$$p_1 = eR[B(R) + \Delta B(R)]$$
$$\Delta P = p_1 - p_0 = eR\Delta B(R) = eR\frac{B_1 - B_0}{2}$$

In order for the electron orbit to be stable, it must be stable against small displacements in the radial and axial directions. Let the magnetic field near the ideal orbit at radius R have components B_z and B_r, though we require $B_r(z=0) = 0$.

(d) Write the equations of motion for the electron in terms of these two components in cylindrical variables.

$$L = K - V$$
$$L = \frac{1}{2}mv^2 = \frac{1}{2}m\left(x_r{}^2 + v_r{}^2\dot{v}_\theta^2 + \dot{v}_z^2\right)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} + F_i$$

$$\vec{F} = e[\vec{v} \times \vec{B}]$$
$$= e[(\dot{x}_z\hat{z} + \dot{x}_r(\sin\theta\hat{y} + \cos\theta\hat{x}) + x_r\dot{x}_\theta(\cos\theta\hat{y} - \sin\theta\hat{x})) \times (B_z\hat{z} + B_r(\sin\theta\hat{y} + \cos\theta\hat{x}))]$$

Now using linearity of the cross-product operator, I get:

$$= e\begin{bmatrix} (-\dot{x}_z B_r \sin\theta + \dot{x}_r B_z \sin\theta + x_r\dot{x}_\theta B_z \cos\theta)\hat{x} \\ +(\dot{x}_z B_r \cos\theta - \dot{x}_r B_z \cos\theta + x_r\dot{x}_\theta B_z \sin\theta)\hat{y} \\ +(-\dot{x}_r B_r \sin\theta\cos\theta + \dot{x}_r B_r \cos\theta\sin\theta - x_r\dot{x}_\theta B_r \cos^2\theta - x_r\dot{x}_\theta B_r \sin^2\theta)\hat{z} \end{bmatrix}$$

Simplifying,

$$= e\begin{bmatrix} (-\dot{x}_z B_r \sin\theta + \dot{x}_r B_z \sin\theta + x_r\dot{x}_\theta B_z \cos\theta)\hat{x} \\ +(\dot{x}_z B_r \cos\theta - \dot{x}_r B_z \cos\theta + x_r\dot{x}_\theta B_z \sin\theta)\hat{y} \\ -(x_r\dot{x}_\theta B_r)\hat{z} \end{bmatrix}$$

and splitting into similar and dissimilar signs for recombination into cylindrical coordinates,

$$= e\begin{bmatrix} (x_r\dot{x}_\theta B_z)(\cos\theta\hat{x} + \sin\theta\hat{y}) \\ +(\dot{x}_z B_r - \dot{x}_r B_z)(-\sin\theta\hat{x} + \cos\theta\hat{y}) \\ -(x_r\dot{x}_\theta B_r)\hat{z} \end{bmatrix} = e\begin{bmatrix} (x_r\dot{x}_\theta B_z)\hat{r} \\ +(\dot{x}_z B_r - \dot{x}_r B_z)\hat{\theta} \\ -(x_r\dot{x}_\theta B_r)\hat{z} \end{bmatrix}$$

Note that:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_r}\right) = \frac{\partial L}{\partial x_r} + F_r$$

$$(take \quad positive \quad e)$$

$$m\ddot{x}_r = mx_r\dot{x}_\theta^2 - ex_r\dot{x}_\theta B_z$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\theta}\right) = \frac{\partial L}{\partial x_\theta} + F_\theta$$

$$\frac{d}{dt}\left[m{x_r}^2\dot{x}_\theta\right] = m\left[2x_r\dot{x}_r\dot{x}_\theta + {x_r}^2\ddot{x}_\theta\right] = -e\left(\dot{x}_z B_r - \dot{x}_r B_z\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_z}\right) = \frac{\partial L}{\partial x_z} + F_z$$

$$m\ddot{x}_z = ex_r\dot{x}_\theta B_r$$

(e) In the vicinity of the orbiting electron at radius R, let the z-component of field be given by $B_z(r) = B_0\left(\frac{r}{R}\right)^{-n}$, where n is the field index with 0 < n < 1. What is the corresponding radial component of field as a function of z?

I see from the equations of motion that in order for this to be orbiting at this radius, I need the following conditions to hold:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = 0$$

$$r\hat{\theta}\left(\frac{\partial}{\partial z}B_r - \frac{\partial}{\partial r}B_z\right) = 0$$

$$\therefore \frac{\partial}{\partial z}B_r - \frac{\partial}{\partial r}B_z = 0 \qquad ?$$

$$\frac{\partial}{\partial z}B_r = \frac{\partial}{\partial r}B_z = -nB_0\frac{1}{R}\left(\frac{r}{R}\right)^{-n-1}$$

$$\therefore B_r = -nB_0\frac{z}{R}\left(\frac{r}{R}\right)^{-n-1}$$

(f) Using suitable linearization of the equations for small displacements, show that the oscillation frequencies of the electron are $\omega_r = (1-n)^{\frac{1}{2}}\omega_0$ and $\omega_z = n^{\frac{1}{2}}\omega_0$ for the radial and axial oscillations, respectively.

Recall that $\dot{x}_\theta = \frac{eB_0}{m}$

$$\ddot{x}_r = x_r\left[\dot{x}_\theta^2 - \frac{e}{m}\dot{x}_\theta B_z\right] = \dot{x}_\theta^2\left[x_r - x_r\left(\frac{x_r}{R}\right)^{-n}\right]$$

$$x_r - x_r\left(\frac{x_r}{R}\right)^{-n} \approx n(x_r - R) + \ldots$$

$$\ddot{x}_r = \dot{x}_\theta^2 n(x_r - R)$$

Now ignoring the inhomogeneous constant portion, I have

$$.\ddot{x}_r = n\dot{x}_\theta^2 x_r$$

$$x \approx A\sinh\left(\sqrt{n\dot{x}_\theta^2}\,t\right) + B\cosh\left(\sqrt{n\dot{x}_\theta^2}\,t\right)$$

This does not yield a periodic function. Something has gone wrong.

For the other mode,

$$m\ddot{x}_z = ex_r\dot{x}_\theta\left[-nB_0\frac{z}{R}\left(\frac{x_r}{R}\right)^{-n-1}\right]$$

$$x_r = R, \dot{x}_\theta = \frac{eB_0}{m}$$

$$\ddot{x}_z = -n\dot{x}_\vartheta^2 x_z$$

$$x_z \approx A\sin\left(\sqrt{n\dot{x}_\theta^2}\,t\right) + B\cos\left(\sqrt{n\dot{x}_\theta^2}\,t\right)$$

$$\omega_z \propto \sqrt{n\dot{x}_\theta^2}, \omega_0 \propto \dot{x}_\theta$$

$$\therefore \omega_z = n^{\frac{1}{2}}\omega_\theta$$