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Classical Electrodynamics Homework 3

1) In the Green's function for the wave equation,

$$G(\vec{x},t;\vec{x}',t')=\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\left(\frac{-c^2}{\omega^2-k^2c^2}\right)$$

suppose we replace the denominator with

$\omega^2-k^2c^2\rightarrow\omega^2-k^2c^2+i\varepsilon$

Compute the new Green's function.

$\tau=t-t' \quad \vec{R}=\vec{x}-\vec{x}'$

$$G(\vec{x},t;\vec{x}',t')=\int\frac{d\omega}{2\pi}e^{-i\omega\tau}\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{R}}\left(\frac{-c^2}{\omega^2-k^2c^2+i\varepsilon}\right)$$

The epsilon essentially tells me how to choose my singularities. In the case in the notes, we chose $\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\left(\frac{-c^2}{(\omega+i\varepsilon)^2-k^2c^2}\right)$. This indicated that omega must be slightly below k^2c^2, and so we got both singularities by shifting $\pm(\omega+i\varepsilon)=kc$. In this case, however, we have $\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\left(\frac{-c^2}{\omega^2-k^2c^2+i\varepsilon}\right)$. This indicates, then, that I will have two singularities $\omega=\pm(kc-i\varepsilon)$. Due to the introduction of the infinitesimal, this contour in w along the real axis will swing upwards along the imaginary axis and go above the positive singularity and below the negative singularity. I will consider the $\tau>0$ and $\tau<0$ cases separately:

$\tau>0$: Close the contour in the upper half-plane, thus taking the singularity $\omega=+(kc-i\varepsilon)$.

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\left(\frac{-c^2}{\omega^2-k^2c^2+i\varepsilon}\right)$$

$$=2\pi i\operatorname{Res}\left\{\frac{e^{-i\omega\tau}}{2\pi}\left(\frac{-c^2}{\omega^2-k^2c^2+i\varepsilon}\right),\omega\rightarrow -kc\right\}=\frac{ice^{ick\tau}}{2k}$$

Now I'd like to get the k into a nicer form:

$$\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{R}}=\int\frac{d^3k}{(2\pi)^3}e^{ikR\cos\theta}=\frac{1}{(2\pi)^2}\int\frac{k}{iR}dk\left(e^{ikR}-e^{-ikR}\right)$$

Recombining this into the original expression, I have

$\tau=t-t' \quad \vec{R}=\vec{x}-\vec{x}'$

$$G(\vec{x},t;\vec{x}',t')=\frac{c\theta(\tau)}{2(2\pi)^2R}\int_0^{\infty}e^{ik(ct+R)}-e^{ik(ct-R)}$$

Now I have using the identity $\int_0^{\infty}e^{ika}=\int_0^{\infty}e^{ik(a+i\varepsilon)}=\frac{i}{a+i\varepsilon}$

$$G(\vec{x},t;\vec{x}',t') = \frac{c\theta(\tau)}{2(2\pi)^2}\left[\frac{i}{R+c\tau} + \frac{i}{R-c\tau}\right]$$

Now consider $\tau < 0$: using

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega\tau}\left(\frac{-c^2}{\omega^2 - k^2c^2 + i\varepsilon}\right)$$

$$= -2\pi i \operatorname{Res}\left\{\frac{e^{-i\omega\tau}}{2\pi}\left(\frac{-c^2}{\omega^2 - k^2c^2 + i\varepsilon}\right), \omega \to kc\right\} = \frac{ice^{-ick\tau}}{2k}$$

where I have been careful to ensure that my contour's direction is counted for, as I closed in the lower half-plane. Now following the same procedure as for the other case, I have

$\tau = t - t' \quad \vec{R} = \vec{x} - \vec{x}'$

$$G(\vec{x},t;\vec{x}',t') = \frac{c[1-\theta(-\tau)]}{2(2\pi)^2 R}\int_0^{\infty} e^{ik(-c\tau+R)} - e^{ik(-c\tau-R)} = \frac{c[1-\theta(-\tau)]}{2(2\pi)^2 R}\left[\frac{i}{R-c\tau} + \frac{i}{R+c\tau}\right]$$

Summing these greater-than and less-than zero portions gives the proper result.

2) **The charge density for a point particle is of the form** $\rho(\vec{x},t) = q\delta^3(\vec{x} - \vec{r}(t))$ **where** $\vec{r}(t)$ **is the trajectory of the projectile. The retarded charge density is then** $\rho\left(\vec{x}, t - \frac{|\vec{x}-\vec{x}'|}{c}\right) = q\delta^3\left[\vec{x} - \vec{r}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right]$**.**

a) **Set the argument of the delta function equal to zero to solve for** $\vec{x}$ **in terms of** $\vec{x}'$ **for the case of linear motion in the z-direction, i.e.** $\vec{r} = (0,0,vt)$**.**

$$0 = \vec{x} - \vec{r}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right) = \vec{x} - v\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\hat{z}$$

Note that here, t refers to the time elapsed at the observer's position, so that $\vec{x} = 0$ when $\frac{|\vec{x}'|}{c} = t$.

Solving the equation $0 = z - vt + \frac{v\sqrt{x'^2 + y'^2 + (z'-z)^2}}{c}$ not surprisingly yields a retarded and an advanced solution in the form of a Lorentz transformation.. I take only the retarded one as a physically valid one.

$$\vec{x}_z = \frac{vt - \frac{v^2}{c^2}\vec{x}'_z}{\left(1-\frac{v^2}{c^2}\right)} - \frac{\sqrt{\left(vt - \frac{v^2}{c^2}\vec{x}'_z\right)^2 - \frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)\left(c^2t^2 - |\vec{x}'|^2\right)}}{\left(1-\frac{v^2}{c^2}\right)}.$$

b) **Find the scalar and vector potential.**

I can just treat the scalar and vector potential as non-retarded, and use the Lorentz-transformed expression above as the source point.

$$\phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(x',t_r)}{R} d^3x' = \frac{1}{4\pi\varepsilon_0}\int \frac{q}{R}\delta^3\left[\vec{x}'-\vec{v}\left(t-\frac{R}{c}\right)\right]d^3x' = \frac{q}{4\pi\varepsilon_0 R}$$

$$R = |\vec{x}-\vec{x}'|$$

$$\vec{y} = \vec{x}'-\vec{v}\left(t-\frac{R}{c}\right)$$

$$d^3x' = \left|\frac{\partial x'_i}{\partial y_j}\right| d^3y \quad \left|\frac{\partial y_j}{\partial x'_i}\right| = \left|\delta_{ij} + \frac{v_i(x_j'-x_j)}{cR}\right| = 1+\left[\frac{\vec{v}}{c}\right]\cdot\left[\frac{\vec{R}}{R}\right]$$

$$\phi = \frac{q}{4\pi\varepsilon_0 R}\frac{1}{\left(1+\left[\frac{\vec{v}}{c}\right]\cdot\left[\frac{\vec{R}}{R}\right]\right)} \quad \vec{A} = \frac{q}{4\pi\varepsilon_0 R}\frac{\vec{v}}{\left(1+\left[\frac{\vec{v}}{c}\right]\cdot\left[\frac{\vec{R}}{R}\right]\right)}$$

3) Maxwell's equations in covariant form are given by

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

If we write

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad A^\mu = \left(\frac{\phi}{c}, \vec{A}\right)$$

we get

$$\left[g_{\mu\nu}\partial^\beta\partial_\beta - \partial_\mu\partial_\nu\right]A^\nu = \mu_0 J_\mu$$

a) Show that the operator $K_{\mu\nu} = g_{\mu\nu}\partial^\beta\partial_\beta - \partial_\mu\partial_\nu$ satisfies the relation for a projection operator, i.e. $K_{\mu\nu}K^{\nu\alpha} = \partial^\beta\partial_\beta K_\mu^\alpha$, meaning that K does not have an inverse.

$$K_{\mu\nu}K^{\nu\alpha}$$
$$= \left[g_{\mu\nu}\partial^\beta\partial_\beta - \partial_\mu\partial_\nu\right]K^{\nu\alpha}$$
$$= \left[g_{\mu\nu}\partial^\beta\partial_\beta - \partial_\mu\partial_\nu\right]g^{A\nu}K_A^\alpha$$
$$= \delta_{\mu A}\partial^\beta\partial_\beta K_A^\alpha - \partial_\mu\partial_\nu g^{A\nu}K_A^\alpha$$
$$= \partial^\beta\partial_\beta K_\mu^\alpha - \partial_\mu\partial_\nu g^{A\nu}g^{B\alpha}K_{AB}$$
$$= \partial^\beta\partial_\beta K_\mu^\alpha - \partial_\mu\partial_\nu g^{A\nu}g^{B\alpha}\left[g_{AB}\partial^\beta\partial_\beta - \partial_A\partial_B\right]$$
$$= \partial^\beta\partial_\beta K_\mu^\alpha - \partial_\mu\partial_\nu g^{B\alpha}\left[\delta_{\nu B}\partial^\beta\partial_\beta - g^{A\nu}\partial_A\partial_B\right]$$
$$= \partial^\beta\partial_\beta K_\mu^\alpha - \partial_\mu\partial_\nu g^{\nu\alpha}\partial^\beta\partial_\beta + \partial_\mu\partial_\nu\partial^\nu\partial_B g^{B\alpha}$$
$$= \partial^\beta\partial_\beta K_\mu^\alpha - \partial_\mu\partial^\alpha\partial^\beta\partial_\beta + \partial_\mu\partial_\nu\partial^\nu\partial^\alpha$$
$$= \partial^\beta\partial_\beta K_\mu^\alpha$$

b) Suppose we modify the wave equations so that

$$\left[g_{\mu\nu}\partial_\beta\partial^\beta - \partial_\mu\partial_\nu(1-\alpha)\right]A^\nu = \mu_0 J_\mu \quad \alpha \quad const$$

and define Green's function by

$$\left[g_{\mu\nu}\partial_\beta\partial^\beta - \partial_\mu\partial_\nu(1-\alpha)\right]G^{\mu\alpha}(\vec{x},t;\vec{x}',t') = g_\nu^\alpha \delta^3(\vec{x}-\vec{x}')\delta(t-t')$$

use the Fourier transform of G

$$G^{\mu\alpha}(\vec{x},t;\vec{x}',t') = \int \frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi} e^{i\vec{k}(\vec{x}-\vec{x}')} e^{i\omega(t-t')} h^{\mu\alpha}(\vec{k},\omega)$$

to find the function for $h^{\mu\alpha}(\vec{k},\omega)$**.**

All I must do is distribute these derivatives inside the integral. This gives:

$$\left[g_{\mu\nu}\left(-\frac{\omega^2}{c^2}-\vec{k}^2\right)+(1-\alpha)\begin{pmatrix}\frac{\omega}{c}\\ \vec{k}\end{pmatrix}_\mu\begin{pmatrix}\frac{\omega}{c}\\ \vec{k}\end{pmatrix}_\nu\right]h^{\mu\alpha}(\vec{k},\omega) = g_\nu^\alpha$$

Shy of actually inverting the multiplying matrix on h, further progression requires a priori knowledge. Take the ansatz

$$h^{\mu\alpha} = a(k^2)g^{\mu\alpha} + b(k^2)k^\mu k^\alpha$$

and substitute this into the expression above. The result implies that

$$a(k^2) = -\frac{1}{k^2}, b(k^2) = \left(1-\frac{1}{\alpha}\right)\frac{1}{k^4}$$