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Classical Electrodynamics 2 Homework 2
Due Feb 2, 2006

1) **The Schrödinger equation in the presence of an electromagnetic field is of the form** $\left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)^2 + q\phi\right]\psi(\vec{x},t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t)$

(a) Show that if the gauge transformation is applied

$$\phi \rightarrow \phi' = \phi - \frac{\partial f}{\partial t}$$

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} f$$

and the transformation of the wave function by

$$\psi \rightarrow \psi' = e^{\frac{iqf}{\hbar}}\psi$$

we have the relation

$$\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}'\right)\psi'(\vec{x},t) = e^{\frac{iqf}{\hbar}}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)\psi(\vec{x},t)$$

Take:

$$\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}'\right)\psi'(\vec{x},t) = \left(\frac{\hbar}{i}\vec{\nabla} - q\left[\vec{A} + \vec{\nabla} f\right]\right)e^{\frac{iqf}{\hbar}}\psi(\vec{x},t) =$$

$$= \frac{\hbar}{i}\left(\psi(\vec{x},t)\vec{\nabla} e^{\frac{iqf}{\hbar}} + e^{\frac{iqf}{\hbar}}\vec{\nabla}\psi(\vec{x},t)\right) - q\left[\vec{A} + \vec{\nabla} f\right]e^{\frac{iqf}{\hbar}}\psi(\vec{x},t) =$$

$$= \frac{\hbar}{i}\left(\psi(\vec{x},t)\frac{iq}{\hbar}e^{\frac{iqf}{\hbar}}\vec{\nabla} f + e^{\frac{iqf}{\hbar}}\vec{\nabla}\psi(\vec{x},t)\right) - q\left[\vec{A} + \vec{\nabla} f\right]e^{\frac{iqf}{\hbar}}\psi(\vec{x},t)$$

$$= qe^{\frac{iqf}{\hbar}}\psi(\vec{x},t)\vec{\nabla} f + \frac{\hbar}{i}e^{\frac{iqf}{\hbar}}\vec{\nabla}\psi(\vec{x},t) - q\vec{A}e^{\frac{iqf}{\hbar}}\psi(\vec{x},t) - qe^{\frac{iqf}{\hbar}}\psi(\vec{x},t)\vec{\nabla} f$$

$$= e^{\frac{iqf}{\hbar}}\left[\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right]\psi(\vec{x},t)$$

(b) Use the relation in (a) to show that the Schrödinger equation remains the same under the transformation given in (a).

Start with the relation

$$\left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}-q\vec{A}'\right)^2+q\phi'\right]\psi'(\vec{x},t)=i\hbar\frac{\partial}{\partial t}\psi'(\vec{x},t)$$

$$\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}-q\vec{A}'\right)^2\psi'(\vec{x},t)+q\left[\phi-\frac{\partial f}{\partial t}\right]e^{\frac{iqf}{\hbar}}\psi(\vec{x},t)=i\hbar\frac{\partial}{\partial t}\left[e^{\frac{iqf}{\hbar}}\psi(\vec{x},t)\right]$$

$$\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}-q\left(\vec{A}+\vec{\nabla}f\right)\right)e^{\frac{iqf}{\hbar}}\left[\frac{\hbar}{i}\vec{\nabla}-q\vec{A}\right]\psi(\vec{x},t)+q\left[\phi-\frac{\partial f}{\partial t}\right]e^{\frac{iqf}{\hbar}}\psi(\vec{x},t)=i\hbar\frac{\partial}{\partial t}\left[e^{\frac{iqf}{\hbar}}\psi(\vec{x},t)\right]$$

$Apply \quad operators \quad and \quad \exp and$

$$\frac{1}{2m}\left(-q\left[\vec{\nabla}f\right]\right)e^{\frac{iqf}{\hbar}}\left[\frac{\hbar}{i}\vec{\nabla}-q\vec{A}\right]\psi(\vec{x},t)+\frac{1}{2m}\frac{\hbar}{i}\left(\frac{iq}{\hbar}e^{\frac{iqf}{\hbar}}\left[\vec{\nabla}f\right]+e^{\frac{iqf}{\hbar}}\vec{\nabla}\right)\left[\frac{\hbar}{i}\vec{\nabla}-q\vec{A}\right]\psi(\vec{x},t)$$

$$-\frac{1}{2m}q\vec{A}e^{\frac{iqf}{\hbar}}\left[\frac{\hbar}{i}\vec{\nabla}-q\vec{A}\right]\psi(\vec{x},t)+q\left[\phi-\frac{\partial f}{\partial t}\right]e^{\frac{iqf}{\hbar}}\psi(\vec{x},t)=i\hbar\left[\frac{iq}{\hbar}e^{\frac{iqf}{\hbar}}\frac{\partial f}{\partial t}+e^{\frac{iqf}{\hbar}}\frac{\partial}{\partial t}\psi(\vec{x},t)\right]$$

$cancel,$

$$\frac{1}{2m}\frac{\hbar}{i}\left(e^{\frac{iqf}{\hbar}}\vec{\nabla}\right)\left[\frac{\hbar}{i}\vec{\nabla}-q\vec{A}\right]\psi(\vec{x},t)-\frac{1}{2m}q\vec{A}e^{\frac{iqf}{\hbar}}\left[\frac{\hbar}{i}\vec{\nabla}-q\vec{A}\right]\psi(\vec{x},t)+q\phi e^{\frac{iqf}{\hbar}}\psi(\vec{x},t)=i\hbar\left[e^{\frac{iqf}{\hbar}}\frac{\partial}{\partial t}\psi(\vec{x},t)\right]$$

$$e^{\frac{iqf}{\hbar}}\left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}-q\vec{A}'\right)^2+q\phi'\right]\psi(\vec{x},t)=i\hbar e^{\frac{iqf}{\hbar}}\frac{\partial}{\partial t}\psi(\vec{x},t)$$

$$\left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}-q\vec{A}'\right)^2+q\phi'\right]\psi(\vec{x},t)=i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t)$$

2 a) From Maxwell's equations derive the wave equation for $\vec{E}$ and $\vec{B}$ fields for the case where ρ and $\vec{J}$ are non-zero.

Let the wave be in a medium with $\vec{D}=\varepsilon\vec{E} \quad \vec{H}=\frac{1}{\mu}\vec{B}$. In this case I have:

$$\vec{\nabla}\times\vec{H}-\frac{\partial\vec{D}}{\partial t}=\vec{J} \quad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0$$

Taking the curl of Faraday's Law, I have

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)+\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)=0$$

Now substituting Ampere's law, I have

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)+\frac{\partial}{\partial t}\mu\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)=0$$

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right)-\vec{\nabla}^2\vec{E}+\frac{\partial}{\partial t}\mu\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)=0$$

Using Gauss's Law, I have $\vec{\nabla} \cdot \vec{D} = \rho$ and so I may substitute

$$\frac{1}{\varepsilon}\vec{\nabla}\rho - \vec{\nabla}^2 \vec{E} + \frac{\partial}{\partial t}\mu\left(\vec{J} + \varepsilon\frac{\partial}{\partial t}\vec{E}\right) = 0 \quad \left[\mu\varepsilon\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right]\vec{E} = -\left[\frac{1}{\varepsilon}\vec{\nabla}\rho + \mu\frac{\partial}{\partial t}\vec{J}\right]$$

Now deriving the analogous equation for $\vec{B}$, I instead take the curl of Ampere's Law:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{H}\right) - \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{D}\right) = \vec{\nabla} \times \vec{J}$$

$$\frac{1}{\mu}\left(\vec{\nabla}\left(\vec{\nabla} \cdot \vec{B}\right) - \vec{\nabla}^2 \vec{B}\right) - \varepsilon\frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right) = \vec{\nabla} \times \vec{J}$$

Now applying no magnetic monopoles, I have

$$-\frac{1}{\mu}\vec{\nabla}^2 \vec{B} + \varepsilon\frac{\partial^2}{\partial t^2}\vec{B} = \vec{\nabla} \times \vec{J} \quad \left[\mu\varepsilon\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right]\vec{B} = \mu\left[\vec{\nabla} \times \vec{J}\right]$$

b) Solve these equations to get $\vec{E}$ and $\vec{B}$ fields in terms of ρ and $\vec{J}$ by using the time dependent Green's function.

Since Green's functions depend only on the combination $\vec{x} - \vec{x}'$, one simply converts $\vec{\nabla}' \to \vec{\nabla}$ and takes it out of the integral:

$$\left[\mu\varepsilon\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right]\vec{E} = -\left[\frac{1}{\varepsilon}\vec{\nabla}\rho + \mu\frac{\partial}{\partial t}\vec{J}\right]$$

$$\vec{E} = -\frac{\mu}{4\pi}\left(\frac{\partial}{\partial t}\right)\int \frac{\vec{J}\left(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c}\right)}{|\vec{x} - \vec{x}'|} d^3x' - \frac{1}{4\pi\varepsilon}\vec{\nabla}\int \frac{\rho\left(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c}\right)}{|\vec{x} - \vec{x}'|} d^3x'$$

$$\left[\mu\varepsilon\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right]\vec{B} = \mu\left[\vec{\nabla} \times \vec{J}\right]$$

$$\vec{B} = \frac{\mu}{4\pi}\vec{\nabla} \times \int \frac{\vec{J}\left(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c}\right)}{|\vec{x} - \vec{x}'|} d^3x'$$

c) Express these wave equations in terms of tensors in Minkowski space.

By making the proper substitutions from Maxwell's laws, I should be able to express these in terms of Minkowski space tensors. Start with the identity:

$$\left[\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}\right] = 0$$

$$\partial^\alpha\left[\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}\right] = 0$$

$$\partial^\alpha \partial_\alpha F_{\beta\gamma} - \mu_0 \partial_\beta J_\gamma + \mu_0 \partial_\gamma J_\beta = 0$$

$$\partial^\alpha \partial_\alpha F_{\beta\gamma} = \mu_0\left[\partial_\beta J_\gamma - \partial_\gamma J_\beta\right]$$

3) The Maxwell's equations can be written in the form

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \quad \mu, \nu = 0,1,2,3$$

where

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ **with** $F^{oi} = -\frac{E^i}{c} \quad F^{ij} = -\varepsilon^{ijk} B^k \quad i, j, k = 1,2,3$

and

$A^\mu = \left(\frac{\phi}{c}, \vec{A} \right)$

Suppose that the spatial dimension is 2 rather than 3, so that $\mu, \nu = 0,1,2$ **and** $i, j = 1,2$**.**

(a) Use the above relation to write down Maxwell's equations in 2 spatial dimensions. The first covariant form is given by

$J^\nu = (c\rho, \vec{J})$ and $\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla} \right) \quad \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla} \right)$

$$\partial_a F^{a0} = \mu_0 J^0$$

$$\frac{1}{c} \vec{\nabla} \cdot \vec{E} = \mu_0 c \rho$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

The above is Gauss's law. There is no reason to believe based on the derivation that this would change based on the number of dimensions.

Now I make an important note: in two dimensions, I cannot ever gain any information about magnetic field from $F^{ij} = -\varepsilon^{ijk} B^k \quad i, j > 0$. This would mean that magnetic fields, if they exist in two dimensions, are never a portion of the calculation. Suppose then that I reduced the size of the antisymmetric tensor so that $F^{ij} = -\varepsilon_{ij} B \quad i, j > 0$. B will now be a constant quantity rather than a vector quantity.

$$\partial_i F^{ij} = \vec{J}^j = \partial_0 F^{0j} + [i > 0] \partial_i F^{ij} = \partial_0 E^j + \partial_i \varepsilon_{ij} B$$

$$\vec{J}^j = \partial_0 E^j + \partial_i \varepsilon_{ij} B$$

Which I identify as

$$\partial_i F^{ij} = \mu_0 \vec{J}^j = \partial_0 F^{0j} + [i > 0] \partial_i F^{ij} = \partial_0 E^j + \partial_i \varepsilon_{ij} B$$

$$\mu_0 \vec{J} = \frac{\partial \vec{E}}{\partial t} + [\partial_x B \hat{y} - \partial_y B \hat{x}]$$

This is, in two dimensions, Ampere's Law.

Now I'd like write an analog of Faraday's Law and the law of no magnetic monopoles, but since the other covariant form is $0 = \varepsilon_{dabc} \partial^a F^{bc}$, I can't gain information from it due to the antisymmetrizer's property that it is zero in a case with duplicated indices!

b) Find the electrostatic potential due to a point charge in 2 dimensions.

In two dimensions, Gauss's Law in integral form must still hold over the two dimensional surface. Then, consider a point charge and a circle about it.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \rightarrow \vec{\nabla} \cdot \vec{E} = \frac{q}{\varepsilon_0}$$

$$\oint_S \vec{\nabla} \cdot \vec{E} = \frac{q}{\varepsilon_0} = 2\pi r E \quad \vec{E} = \hat{r}\frac{q}{2\pi r \varepsilon_0} \quad \phi = -\frac{q}{\varepsilon_0}\ln r$$

The justification for this surface integration is solved as a problem on the first test in EM 1.

c) Use the Lorentz transformation to find the relations between electromagnetic fields in different inertial frames in 2 dimensions.

The Lorentz transformation would look like:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in two dimensions, where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad \beta = \frac{v}{c}$

so that when the frame accelerates in the x direction at velocity v, so that using

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} \\ \frac{E_x}{c} & 0 & -B \\ \frac{E_y}{c} & B & 0 \end{pmatrix} \quad \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E'_x}{c} & -\frac{E'_y}{c} \\ \frac{E'_x}{c} & 0 & -B \\ \frac{E'_y}{c} & B & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{E_x}{c} & -\beta B\gamma - \frac{E_y\gamma}{c} \\ \frac{E_x}{c} & 0 & B\gamma + \beta\frac{E_y\gamma}{c} \\ \beta B\gamma + \frac{E_y\gamma}{c} & -B\gamma - \beta\frac{E_y\gamma}{c} & 0 \end{pmatrix}$$

So you see now why it was necessary to say that the magnetic field, while having no coordinate meaning, might well have been a constant (indeed, the nature of the definition of Maxwell's laws in covariant form using a derivative left open the option of a globally constant magnetic field). It would play no part in Maxwell's laws as they stand in two dimensions, but would appear and be measurable only as an anomaly in the electric fields in fast-moving frames of reference.

d) Find the Lorentz force acting on a charged particle in 2 dimensions.

Suppose that in this two dimensional world, it is in fact a 2D slice of a 3D world where one dimension is uniform. In that case, $B \to B\hat{z}$, and so

$$\vec{F} = q\left(F^{i0}u_0 + F^{ij}u_j\right) = q\gamma\left(\vec{E} + B\left(\vec{v} \times \hat{z}\right)\right)$$