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Quantum Mechanics
Problem Source: **CMU August 2006 Qualifying Exam**

A Stern-Gerlach experiment that analyzes spin along the i-direction (SG; i) is characterized by the Hamiltonian $H_i = \mu N S_i$, where S_i is the i-th component of the $\vec{S}$ spin operator.

(a) Consider spin-1 particles and use the following basis of S_z eigenstates: $\{|1\rangle, |0\rangle, |1\rangle\}$. What eigenvalue(s) does $\vec{S}^2$ have?

$$S^2|s \quad m\rangle = \hbar^2 s(s+1)|s \quad m\rangle$$
$$\therefore$$
$$2\hbar^2$$

(b) Using the S_i commutation relations, derive raising and lowering operators, $S_\pm$, which obey the commutation relations $[S_z, S_\pm] = \pm\hbar S_\pm \quad [S_+, S_-] = 2\hbar S_z \quad S_+ = S_-^+$.

Recall: $[S_x, S_y] = i\hbar S_z \quad [S_y, S_z] = i\hbar S_x \quad [S_z, S_x] = i\hbar S_y$

Guess: $S_+ = S_x + iS_y \quad S_- = S_x - iS_y$

Verify:
$$[S_z, S_\pm] = \pm\hbar S_\pm$$
$$[S_z, S_+] = [S_z, S_x + iS_y] = \hbar(iS_y + S_x) = \hbar S_+$$
$$[S_z, S_-] = [S_z, S_x - iS_y] = \hbar(iS_y - S_x) = -\hbar S_-$$
$$[S_+, S_-] = [S_x + iS_y, S_x - iS_y] = i[S_y, S_x] - i[S_x, S_y] = 2\hbar S_z$$
$$Clearly, S_+ = S_-^+$$

(c) Assuming that m takes on a value such that $S_\pm|m\rangle \neq 0$, show that $S_\pm|m\rangle = N_m|m \pm 1\rangle$ and determine the proportionality constant N_m.

Note: $S^2 = S_z^2 + S_y^2 + S_x^2$

$$S_+S_- = (S_x + iS_y)(S_x - iS_y) = S_x^2 + S_y^2 + iS_yS_x - iS_xS_y$$
$$= S_x^2 + S_y^2 + i(S_yS_x - S_xS_y) = S_x^2 + S_y^2 + i(-i\hbar S_z) = S_x^2 + S_y^2 + \hbar S_z$$

Similarly,

$$S_-S_+ = S_x^2 + S_y^2 - \hbar S_z$$

Then,

$$S^2 = S_z^2 + S_+S_- - \hbar S_z = S_z^2 + S_-S_+ + \hbar S_z$$

From part (b), it's clear that if $S_z|m\rangle = m\hbar|m\rangle$, then

$$S_zS_\pm|m\rangle = S_\pm S_z|m\rangle + [S_z, S_\pm]|m\rangle = m\hbar S_\pm|m\rangle \pm \hbar S_\pm|m\rangle = (m \pm 1)\hbar S_\pm|m\rangle,$$

justifying that $S_\pm|m\rangle$ does in fact raise or lower the ket.

Determining the proportionality constant, take:

$$S^2|m\rangle = 2\hbar^2|m\rangle = m^2\hbar^2|m\rangle + S_+S_-|m\rangle - m\hbar^2|m\rangle$$
$$\therefore S_+S_-|m\rangle = (2\hbar^2 - m^2\hbar^2 + m\hbar^2)|m\rangle$$
$$\therefore S_-S_+|m\rangle = (2\hbar^2 - m^2\hbar^2 - m\hbar^2)|m\rangle$$

This indicates that the normalization must be $N_m = \hbar\sqrt{2 - m(m \pm 1)}$.

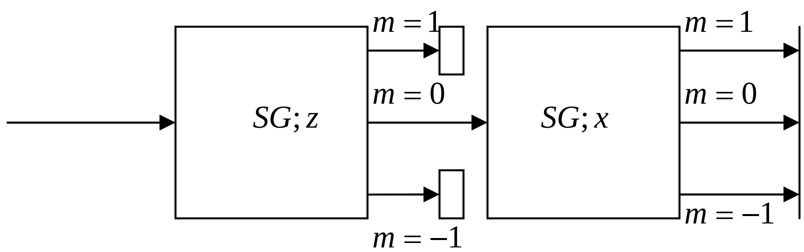
Since at the top and bottom of the ladder this must vanish, then,

$$S_\pm = N_m|m \pm 1\rangle = \hbar\sqrt{2 - m(m \pm 1)}|m \pm 1\rangle$$

(d) What is the 3×3 matrix for S_x in this basis?

$$S_x = \frac{S_+ + S_-}{2}$$
$$S_x|-1\rangle = \frac{S_+ + S_-}{2}|-1\rangle = \frac{1}{2}S_+|-1\rangle = \frac{\hbar}{\sqrt{2}}|0\rangle$$
$$S_x|0\rangle = \frac{S_+ + S_-}{2}|0\rangle = \frac{\hbar}{\sqrt{2}}|-1\rangle + \frac{\hbar}{\sqrt{2}}|1\rangle$$
$$S_x|1\rangle = \frac{S_+ + S_-}{2}|1\rangle = \frac{\hbar}{\sqrt{2}}|0\rangle$$
$$S_x = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad basis: \begin{bmatrix} |-1\rangle \\ |0\rangle \\ |1\rangle \end{bmatrix}$$

(e) A beam of spin-1 particles, with random spin orientation, enters a $(SG;z)$ apparatus. The $m=\pm 1$ states are blocked, with $m=0$ transmitted to a second $(SG;x)$ apparatus. What are the relative intensities for the beams measured at the screen?



First, I need the eigenvectors of S_x.

$$\det\begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} = -\lambda(\lambda^2-1)+\lambda = -\lambda^3+2\lambda = \lambda(2-\lambda^2) = \lambda(\sqrt{2}-\lambda)(\sqrt{2}+\lambda)$$

$$null\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = null\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} x=-z \\ y=0 \\ z \end{bmatrix} \rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : |0_x\rangle \quad m_x = 0$$

$$null\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} = null\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} x=z \\ y=\sqrt{2}z \\ z \end{bmatrix} \rightarrow \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} : |1_x\rangle \quad m_x = 1$$

$$null\begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} = null\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} x=z \\ y=-\sqrt{2}z \\ z \end{bmatrix} \rightarrow \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} : |-1_x\rangle \quad m_x = -1$$

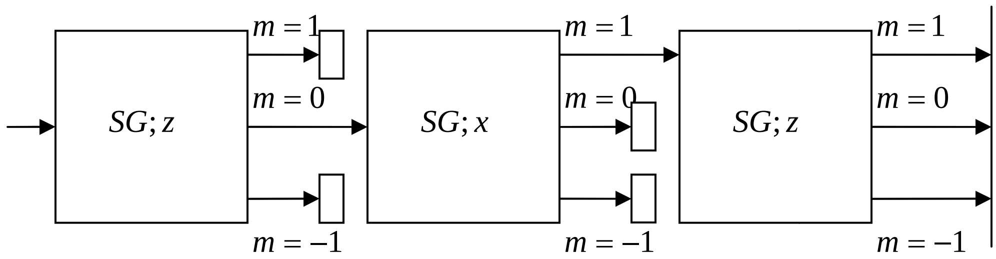
Now,

$$|0\rangle\langle 0\|1_x\rangle\langle 1_x| \propto \frac{1}{\sqrt{2}} \quad |0\rangle\langle 0\|0_x\rangle\langle 0_x| = 0 \quad |0\rangle\langle 0\|-1_x\rangle\langle -1_x| \propto -\frac{1}{\sqrt{2}}$$

Or, since intensity is proportional to square of magnitude,

$$\frac{I_1}{I_{tot}} = \frac{1}{2} \quad \frac{I_2}{I_{tot}} = 0 \quad \frac{I_3}{I_{tot}} = \frac{1}{2}$$

(f) Now suppose that the $m=1$ state is transmitted to another $(SG;z)$ apparatus as shown below. What is the relative intensity of the beams measured in this case?



$$|1_x\rangle\langle 1_x \| 1\rangle\langle 1| \propto \frac{1}{2} \quad |1_x\rangle\langle 1_x \| 0\rangle\langle 0| \propto \frac{\sqrt{2}}{2} \quad |1_x\rangle\langle 1_x \| 1\rangle\langle 1| \propto \frac{1}{2}$$

Or, since intensity is proportional to square of magnitude,

$$\frac{I_1}{I_{tot}} = \frac{1}{4} \quad \frac{I_2}{I_{tot}} = \frac{1}{2} \quad \frac{I_3}{I_{tot}} = \frac{1}{4}$$