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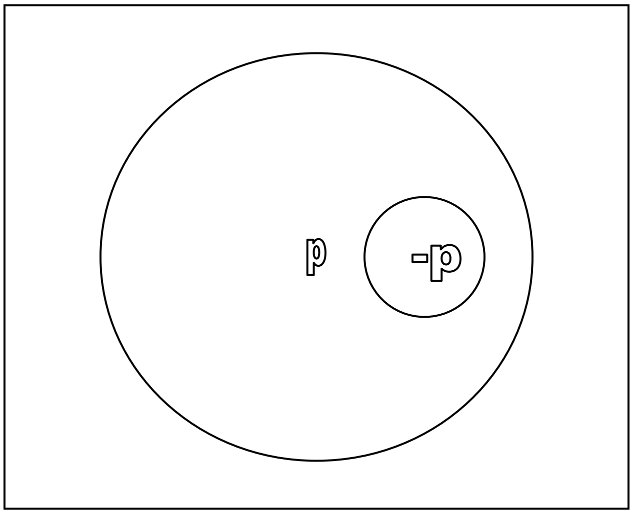
Classical Electrodynamics I Final
Homework for Classical Electrodynamics II

1 a) A sphere of radius R_1 has a small hollow spherical region of radius R_2 located at a distance a from the center of the large sphere. There is a uniform charge density ρ throughout the volume. Compute the electric field in the hollow region.

Use the principle of superposition to turn this into two spheres of uniform charge:



Now using Gauss's law on each, I find that in the hollow region the field will be the sum of the field from the large sphere L and the small sphere S.

$$\vec{E}_L(\vec{r}) = \frac{1}{3\varepsilon_0}\rho r_L \hat{r}_L \qquad \vec{E}_S(\vec{r}) = \frac{-1}{3\varepsilon_0}\rho r_S \hat{r}_S$$

$$\vec{E} = \vec{E}_L(\vec{r}) + \vec{E}_S(\vec{r}) = \frac{\rho}{3\varepsilon_0}\left(r_L \hat{r}_L - r_S \hat{r}_S\right)$$

Letting the center of the large sphere be the origin and the origin of the small sphere be at some vector $\vec{a}$.

$$\vec{E} = \frac{\rho}{3\varepsilon_0}\left(r\hat{r} - \left(\vec{r} - \vec{a}\right)\right) = \frac{\rho}{3\varepsilon_0}\vec{a}$$

1 b) Consider a sphere of radius R centered at the origin. Suppose a point charge q is put at the origin and that is the only charge inside and outside the sphere. Furthermore, the potential is $\phi = V_0 \cos\theta$ on the surface of the sphere. Find the potential both inside and outside the sphere.

The solutions to the Laplacian must take the form of spherical harmonics:

$$\phi(r,\theta,\phi) = \sum_{l,m}\left(A_{l,m} r^{l} + \frac{B_{l,m}}{r^{l+1}} \right) Y_{l,m}(\theta,\phi)$$

Due to the azimuthal symmetry, however, I have

$$\phi(r,\theta) = \sum_{l}\left(A_l r^{l} + \frac{B_l}{r^{l+1}} \right) Y_{l,0}(\theta).$$

Now my constraints are as follows: outside the sphere, $\begin{matrix} \phi(\infty,\theta) \to 0 \\ \phi(R,\theta) \to V_0 \cos\theta \end{matrix}$. Inside the sphere, $\begin{matrix} \phi(R,\theta) \to V_0 \cos\theta \\ \nabla^2 \phi = \dfrac{q}{\varepsilon_0} \delta^3(\vec{r}) \end{matrix}$.

I recall that $\nabla^2 \frac{1}{4\pi r} = \delta^3(\vec{r})$, so now I have inside:

$$\phi(r,\theta) = A_0 \frac{1}{2\sqrt{\pi}} + \frac{B_0}{r}\frac{1}{2\sqrt{\pi}} + A_1 r \sqrt{\frac{3}{4\pi}} \cos(\theta)$$

and outside,

$$\phi(r,\theta) = \frac{B_1}{r^2}\sqrt{\frac{3}{4\pi}} \cos(\theta).$$

Now fixing my constants, I need

$$\frac{B_0}{r}\frac{1}{2\sqrt{\pi}} = \frac{1}{4\pi r} \quad B_0 = \frac{1}{\sqrt{4\pi}}$$

$$A_0 \frac{1}{2\sqrt{\pi}} = \frac{B_0}{R}\frac{1}{2\sqrt{\pi}} \quad A_0 = \frac{1}{\sqrt{4\pi} R}\frac{q}{\varepsilon_0}$$

$$A_1 R \sqrt{\frac{3}{4\pi}} \cos(\theta) = \frac{B_1}{R^2}\sqrt{\frac{3}{4\pi}} \cos(\theta) = V_0 \cos\theta$$

$$A_1 R = \frac{B_1}{R^2} = \sqrt{\frac{4\pi}{3}} V_0 \quad A_1 = \frac{1}{R}\sqrt{\frac{4\pi}{3}} V_0 \quad B_1 = R^2 \sqrt{\frac{4\pi}{3}} V_0$$

which substituting into the forms above gives the potential satisfying these constraints.

2 a) A spherical shell of radius a has a surface charge density given in spherical coordinates by $\sigma = \sigma_0 \sin\theta \cos\phi$, where σ_0 is some constant and the origin is at the center of the spherical shell. Find

$$Q = \int d^3x\rho \quad \vec{P} = \int d^3x\rho\vec{x} \quad Q_{i,j} = \int d^3x\rho\left(3x_i x_j - \delta_{i,j}x^2\right)$$

Notice that in spherical coordinates, $x = r\sin\theta\cos\phi$.

$$Q = \int d^3x\rho = \int d^3x\frac{1}{r}x\delta(x-a) = 0 \quad by \quad symmetry$$

$$\vec{P} = \int d^3x\rho\vec{x}$$

$$P_x = \int d^3xx^2\frac{1}{r}\delta(r-a) = \sigma_0 a^3\int \sin^3\theta\cos^2\phi d\theta d\phi = \frac{4\pi}{3}\sigma_0 a^3$$

$$P_y, P_z = 0 \quad \text{int}egral \quad over \quad odd \quad functions$$

$$Q_{i,j} = 0$$

The quadrupole terms must all die since the delta term will always contain an odd number of x factors, and the first term must always contain an odd number of some term and so the zero-centered integral over it will always disappear. The dipole moments in the y and z directions have been simplified to zero by a similar argument, since the integral of an odd function centered at zero must always be zero.

2 b) Let $\vec{B}$ be the magnetic field produced by some steady current density,

$$\vec{B} = \vec{\nabla}\times\vec{A}, \quad \vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'.$$

Suppose we average $\vec{B}$ over a sphere containing all the current density,

$\vec{B}_{avg} = \dfrac{1}{\frac{4\pi}{3}a^3}\int\vec{B}d^3x$ **, where a is the radius of the sphere. Show that**

$\vec{B}_{avg} = \dfrac{\mu_0}{4\pi}\dfrac{2\vec{m}}{a^3}$ **where $\vec{m}$ is the magnetic moment of the current density.**

$$\frac{1}{2}\int\left(\vec{J}(\vec{x}',t)\times\vec{x}'\right)d^3x'=\vec{m}$$

$$\vec{B}=\vec{\nabla}\times\vec{A}$$

$$\vec{B}_{avg}=\frac{1}{\frac{4\pi}{3}a^3}\int\vec{B}d^3x=\frac{1}{\frac{4\pi}{3}a^3}\int\vec{\nabla}\times\vec{A}d^3x=\frac{-1}{\frac{4\pi}{3}a^3}\oint\vec{A}\times d\vec{s}$$

$$=\frac{-1}{\frac{4\pi}{3}a^3}\oint\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'\times\hat{r}a^2d\Omega$$

$$Since\quad\int\frac{\hat{r}d\Omega}{|\vec{x}-\vec{x}'|}=\frac{4\pi}{3a^2}\vec{x}'$$

$$=\frac{-1}{\frac{4\pi}{3}a^3}\frac{\mu_0}{4\pi}\frac{4\pi}{3a^2}\left[-2a^2\int\frac{1}{2}\left(\vec{x}'\times\vec{J}(\vec{x}',t)\right)d^3x'\right]=\frac{\mu_0}{4\pi}\frac{2\vec{m}}{a^3}$$

3 a) Some dielectric material under some condition will have the property that the displacement $\vec{D}$ and the electric field $\vec{E}$ are related by $\vec{D}=\varepsilon_0\vec{E}+i\vec{E}\times\vec{g}$ where $\vec{g}$ is a constant real vector. Consider plane waves propagating in the direction of $\vec{g}$ taken to be in the z-direction, $\vec{E}(\vec{x},t)=\vec{E}_0e^{i(\vec{k}\cdot\vec{x}-\omega t)}$**. Starting from Maxwell's equations find the possible values for the index of refraction, $n\equiv\frac{kc}{\omega}$, in terms of ε_0 and g.**

The relevant Maxwell equations are

$$\begin{aligned}&\frac{1}{\mu_0}\vec{\nabla}\times\vec{B}-\frac{\partial}{\partial t}\vec{D}=0\\&\vec{\nabla}\times\vec{E}+\frac{\partial}{\partial t}\vec{B}=0\end{aligned}\quad.$$

Taking del-cross the second and substituting the first, I have:

$$\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial}{\partial t} \vec{D}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) + \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right) = 0 \quad \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \vec{\nabla}^2 \vec{E} + \frac{\partial}{\partial t}\left(\mu_0 \frac{\partial}{\partial t} \vec{D}\right) = 0$$

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \vec{\nabla}^2 \vec{E} + \mu_0 \frac{\partial^2}{\partial t^2}\left[\varepsilon_0 \vec{E} + i\vec{E} \times \vec{g}\right] = 0$$

Now I just need to impose the simplifying constraints. I choose $\vec{k}$ and $\vec{g}$ to be in the z-direction as per the problem constraints, and so $\vec{E}_0 = a\hat{x} + b\hat{y}$. Now I have that

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}_0 e^{i(kz - \omega t)}\right) = 0$$

$$\vec{\nabla}^2 \vec{E} = -\vec{E}_0 k^2 e^{i(kz - \omega t)}$$

$$\frac{\partial^2}{\partial t^2} \vec{E} = -\vec{E}_0 \omega^2 e^{i(kz - \omega t)}$$

Giving

$$-k^2 \vec{E} + \mu_0 \omega^2 \left[\varepsilon_0 \vec{E} + i\vec{E} \times \vec{g}\right] = 0$$

Now choose $\vec{E}_0 = a\hat{x} + b\hat{y}$

$$-k^2 (a\hat{x} + b\hat{y}) + \mu_0 \omega^2 \left[\varepsilon_0 (a\hat{x} + b\hat{y}) + i(a\hat{x} + b\hat{y}) \times g\hat{z}\right] = 0$$

$$-k^2 (a\hat{x} + b\hat{y}) + \mu_0 \omega^2 \left[\varepsilon_0 (a\hat{x} + b\hat{y}) + ig(b\hat{x} - a\hat{y})\right] = 0$$

$$-k^2 a + \mu_0 \omega^2 \varepsilon_0 a + i\mu_0 \omega^2 gb = 0$$

$$-k^2 b + \mu_0 \omega^2 \varepsilon_0 b - i\mu_0 \omega^2 ga = 0$$

Solving this inhomogeneous system while eliminating a and b, I get

$$-k^2 + \mu_0 \omega^2 \left(\varepsilon_0 + \frac{g^2 \mu_0 \omega^2}{k^2 - \varepsilon_0 \mu_0 \omega^2}\right) = 0$$

$$k^2 = \mu_0 \omega^2 (\varepsilon_0 \pm g)$$

Since $n \equiv \frac{kc}{\omega}$, this constrains that

$$\frac{kc}{\omega} = \pm \frac{c}{\omega} \sqrt{\mu_0 \omega^2 (\varepsilon_0 \pm g)} = \pm c\sqrt{\mu_0 (\varepsilon_0 \pm g)}$$

3 b) The retarded potentials are of the form:

$$\phi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho\left(\vec{x}',t-\frac{R}{c}\right)}{R}d^3\vec{x}' \quad R=|\vec{x}-\vec{x}'|$$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int\frac{\vec{J}\left(\vec{x}',t-\frac{R}{c}\right)}{R}d^3\vec{x}'$$

In the radiation zone where r>>d, d is the size of the charge or current density distribution. Show that to first order in $\frac{d}{r}$, we can approximate these potentials as

$$\phi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r}+\frac{\hat{x}\cdot\vec{p}(t_0)}{r^2}+\frac{\hat{x}\cdot\dot{\vec{p}}(t_0)}{cr}+....\right] \quad t_0=t-\frac{r}{c}$$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\frac{\dot{\vec{p}}(t_0)}{r}+...$$

where Q and $\vec{p}$ are total charge and dipole moment respectively.

First, note that

$$\frac{1}{R}=\frac{1}{r}+\frac{\vec{r}\cdot\vec{r}'}{r^3}+...$$
$$R=r-\frac{\vec{r}\cdot\vec{r}'}{r}+...$$

Substituting this approximation, I have

$$\phi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\int\left(\frac{1}{r}+\frac{\vec{r}\cdot\vec{r}'}{r^3}\right)\rho\left(\vec{x}',t-\frac{1}{c}\left(r-\frac{\vec{r}\cdot\vec{r}'}{r}\right)\right)d^3\vec{x}' \quad R=|\vec{x}-\vec{x}'|$$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int\left(\frac{1}{r}+\frac{\vec{r}\cdot\vec{r}'}{r^3}\right)\vec{J}\left(\vec{x}',t-\frac{1}{c}\left(r-\frac{\vec{r}\cdot\vec{r}'}{r}\right)\right)d^3\vec{x}'$$

First I'll focus on the potential expression. I notice that by taking the $\frac{1}{r}$ term only and imposing conservation of charge, I get

$$\phi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\int\left(\frac{1}{r}\rho\left(\vec{x}',t-\frac{R}{c}\right)+\frac{\vec{x}\cdot\vec{x}'}{r^3}\rho\left(\vec{x}',t-\frac{R}{c}\right)\right)d^3\vec{x}'$$

$$\phi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r}\right]+\frac{\vec{x}}{r^3}\int\vec{x}'\rho\left(\vec{x}',t-\frac{1}{c}\left(r-\frac{\vec{x}\cdot\vec{x}'}{rc}\right)\right)d^3\vec{x}'$$

$$\phi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r}\right]+\frac{\vec{x}}{r^3}\left[\int\vec{x}'\rho\left(\vec{x}',t-\frac{r}{c}\right)+\frac{\vec{x}\cdot\vec{x}'}{rc}\dot{\rho}\left(\vec{x}',t-\frac{r}{c}\right)\right]+...d^3\vec{x}'$$

$$\phi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r}\right]+\frac{\vec{x}}{r^3}\left[\int\vec{x}'\rho\left(\vec{x}',t-\frac{r}{c}\right)+\frac{r}{c}\frac{d}{r}\dot{\rho}\left(\vec{x}',t-\frac{r}{c}\right)\right]+...d^3\vec{x}'$$

$$\phi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r}\right]+\frac{\vec{x}}{r^3}\vec{p}\left(\vec{x}',t-\frac{r}{c}\right)+\frac{\vec{x}}{r^2c}\dot{\vec{p}}\left(\vec{x}',t-\frac{r}{c}\right)$$

To the first order, I have

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi r}\int\vec{J}\left(\vec{x}',t-\frac{R}{c}\right)d^3\vec{x}'$$

Or

$$\vec{A}(\vec{x},t)=-\frac{\mu_0}{4\pi r}\int\vec{x}'\cdot\left(\vec{\nabla}\cdot\vec{J}\left(\vec{x}',t-\frac{R}{c}\right)\right)d^3\vec{x}'$$

$$\vec{A}(\vec{x},t)=-\frac{\mu_0}{4\pi r}\int\vec{x}'\cdot\dot{\rho}\left(\vec{x}',t-\frac{R}{c}\right)d^3\vec{x}'$$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi r}\dot{\vec{p}}\left(\vec{x}',t-\frac{R}{c}\right)$$

For the final simplification I have taken the dipole moment of the charge density's time derivative. Since these derivatives commute, it's the same as the time derivative of the dipole moment.

4 a) Let x^μ and u^μ be the coordinates and the 4-velocity of a charged particle. Define the vector r^μ by $r^\mu = x^\mu - \frac{1}{c^2}(u\cdot x)u^\mu$. Show that r^μ is space-like, i.e. $r^\mu r_\mu < 0$.

Note that $u\cdot u = c^2$. Now $r^\mu = x^\mu - \frac{1}{c^2}(u\cdot x)u^\mu$ is x^μ minus its projection into the time-like u^μ space. Since the time-like element is thus removed, the only part left can be space-like and $r^\mu r_\mu < 0$.

$x^\mu = (ct,\vec{x}) \quad u^\mu = \lambda(c,\vec{v})$

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4 b) Let $r=\sqrt{r^{\mu}r_{\mu}}$ **and consider the 4-vector potential** A^{μ} **of the form** $A^{\mu}=\frac{q\mu^{\mu}}{4\pi\varepsilon_0 r}$. **Compute the electric and magnetic fields from 4-vector potential** A^{μ}.

$$F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}$$

$$F^{\mu\nu}=\begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{bmatrix}$$

$$\vec{E}_x=\partial^{x}A^{0}-\partial^{0}A^{x}$$

$$\partial^{0}=-\partial_0 \quad \partial^{c}=\partial_c$$

$$\partial_0=\frac{1}{c}\frac{\partial}{\partial t} \quad \partial_c=\frac{\partial}{\partial c}$$

$$x^{\mu}=(ct,\vec{x}) \quad u^{\mu}=\lambda(c,\vec{v})$$

$$\vec{E}_c=\partial_c A^{0}-\partial_0 A^{c}$$

$$\vec{E}_c=\frac{1}{4\pi\varepsilon_0}\left(\partial_c\frac{qu^{0}}{4\pi\varepsilon_0 r}-\partial_0\frac{qu^{c}}{4\pi\varepsilon_0 r}\right)=\frac{1}{4\pi\varepsilon_0}\left(\partial_c\frac{\lambda qc}{4\pi\varepsilon_0 r^{2}}-\partial_0\frac{qu^{c}}{4\pi\varepsilon_0 r}\right)$$

I also note that the scalar product in the denominator is invariant under Lorentz transformation. As usual, velocity must be the time derivative of position, e.g. $\partial_0 x^{i}=u^{i}$

$$\frac{1}{r}=\frac{1}{\sqrt{r^{\mu}r_{\mu}}}$$

$$\partial^{\mu}\frac{1}{r}=-\frac{\left(-1-\frac{1}{c^{2}}u^{0}u_{0}\right)u^{\mu}}{r^{2}}=\frac{\left(1+\frac{1}{c^{2}}u^{0}u_{0}\right)u^{\mu}}{r^{2}}$$

$$\partial_c\frac{1}{r}=-\left(-\frac{1-\frac{1}{c^{2}}u^{c}x_{c}u^{\mu}}{r^{2}}\right)$$

Notice the extra – sign from the covariant dot product.

Now by substituting these into the formula above, it is not difficult to get the fields:

$$\vec{E}_c = \frac{q}{4c\pi\varepsilon_0 r^2}\left(\left(1-\frac{1}{c^2}u^c x_c\right)u^0 - \left(1+\frac{1}{c^2}u^0 u_0\right)u^c\right)$$

$$\vec{B}_c = \frac{q}{4\pi\varepsilon_0 r^2}\left(\left(1-\frac{1}{c^2}u^{c-1}u_{c-1}\right)u^{c+1} - \left(1-\frac{1}{c^2}u^{c+1}u_{c+1}\right)u^{c-1}\right)$$

Where above I take the index to mean a cyclic value of {x, y, z}.

4 c) For the special case where the charged particle is at rest, what are the $\vec{E}$ and $\vec{B}$ fields from this formula.

Set $u = (c,0,0,0)$.

Now clearly both of the terms in the magnetic field above vanish since all nonzero-indexed values of u are zero.

$$\vec{E}_c = \frac{q}{4\pi\varepsilon_0 r^2}\left(\left(1-\frac{1}{c^2}u^c x_c\right)u^0 - \left(1+\frac{1}{c^2}u^0 u_0\right)u^c\right)$$

$$\rightarrow \frac{q}{4\pi\varepsilon_0 r^2 c}(c) = \frac{q}{4\pi\varepsilon_0 r^2}$$